1.) $G = SU_L(4) \times SU_R(4)$; $H = SU_V(4)$

$\rightsquigarrow n_G = d(G) - d(H) = n_f^2 - 1 = 15.$

2.)
$$\mathcal{L} = \frac{F^2}{4} \langle \partial_\mu U^\dagger \partial^\mu U \rangle$$

set $F = 1$ for the moment

a.) $U(x) = \exp(i \, \vec{\sigma} \vec{\pi})$

$$= \underbrace{\cos(|\vec{\pi}|)} + i \, \vec{\pi} \cdot \vec{\sigma} \cdot \underbrace{\frac{1}{|\vec{\pi}|} \sin(|\vec{\pi}|)}_{(1 - \vec{\pi}^2/6 + \dots)}$$

$$= 1 + i \vec{\pi} \vec{\sigma} - \frac{\vec{\pi}^2}{2} - i \vec{\pi} \cdot \sigma \cdot \vec{\pi}^2/6$$

$$\partial_\mu U = i \, \partial_\mu \vec{\pi} \cdot \vec{\sigma} - \vec{\pi} \partial_\mu \vec{\pi} - i \, \partial_\mu \vec{\pi} \cdot \vec{\sigma} \, \vec{\pi}^2/6$$

$$- i \, \vec{\pi} \cdot \vec{\sigma} \, \frac{1}{3} \vec{\pi} \partial_\mu \vec{\pi}$$

$$\langle \partial_\mu U \partial^\mu U^\dagger \rangle = \Big\langle \partial_\mu \vec{\pi}\cdot\vec{\sigma}\ \partial^\mu \vec{\pi}\cdot\vec{\sigma} + (\vec{\pi}\,\partial_\mu\vec{\pi})^2$$
$$- 2\, \partial_\mu \vec{\pi}\cdot\vec{\sigma}\ \partial^\mu \vec{\pi}\cdot\vec{\sigma}\ \cdot \vec{\pi}^2/6$$
$$- 2\, \partial_\mu \vec{\pi}\cdot\vec{\sigma}\ \vec{\pi}\cdot\vec{\sigma}\ \tfrac{1}{3}\, \vec{\pi}\,\partial^\mu\vec{\pi} \Big\rangle$$

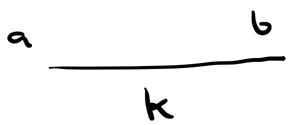
Note : $\langle \sigma^a \sigma^b \rangle = \frac{1}{2}\langle \{\sigma^a, \sigma^b\} \rangle = \langle \mathbb{1} \rangle \delta^{ab} = 2\delta^{ab}$

$$\rightsquigarrow \langle \partial_\mu U \partial^\mu U^\dagger \rangle = 2\, \partial_\mu \vec{\pi}\,\partial^\mu \vec{\pi} + 2(\vec{\pi}\,\partial_\mu \vec{\pi})^2$$
$$- \frac{2}{3}\, \partial_\mu \vec{\pi}\,\partial^\mu \vec{\pi}\cdot \vec{\pi}^2 - \frac{4}{3}(\vec{\pi}\,\partial_\mu\vec{\pi})^2$$

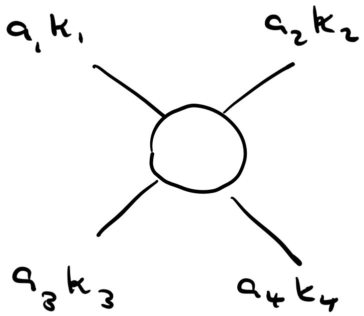
Restoring the factors of F, we get the final result

$$\frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger \rangle = \frac{1}{2}\, \partial_\mu \vec{\pi}\,\partial^\mu \vec{\pi} + \frac{1}{6F^2}(\vec{\pi}\,\partial_\mu \vec{\pi})^2$$
$$- \frac{1}{6F^2}\, \partial_\mu \vec{\pi}\,\partial^\mu \vec{\pi}\cdot \vec{\pi}^2$$

Feynman rules:



$$= \frac{i}{k^2}\delta^{ab}$$



$$\hat{=} \frac{i}{6F^2}\Big\{ -\delta^{a_1a_2}\,\delta^{a_3a_4}\,k_2\cdot k_4 + \delta^{a_1a_2}\,\delta^{a_3a_4}\,k_1\cdot k_2 + \text{"permutations"} \Big\}$$

$i\partial_\mu \hat{=} k$

$$\hat{=} \frac{i}{3F^2}\Big\{ \delta^{a_1a_2}\,\delta^{a_3a_4}\Big(\overset{s}{2k_1\cdot k_2} + \overset{s}{2k_3\cdot k_4} - \overset{t/2}{k_1\cdot k_3} - \overset{u/2}{k_1\cdot k_4} - \overset{u/2}{k_2\cdot k_3} - \overset{t/2}{k_2\cdot k_4}\Big) + \text{"}2\leftrightarrow3\text{"} + \text{"}2\leftrightarrow4\text{"}\Big\}$$

on shell:

$\rightsquigarrow \; 2s - t - u = 3s$

c.)

To get the scattering amplitude, it is good enough to know the on-shell vertex and of course we can always use momentum conservation $\sum_i k_i = 0$.

With this, we have $\quad s=(p_1+p_2)^2$

$$\hat{=} \frac{i}{F^2}\left\{ \delta^{a_1a_2}\delta^{a_3a_4}\, s + \text{“}2 \leftrightarrow 3\text{”} + \text{“}2 \leftrightarrow 4\text{”} \right\}$$

$$+ \text{“off-shell} \propto p_i^2\text{”}$$

$t=(p_1+p_3)^2 \qquad u=(p_1+p_4)^2$

$$\hat{=} \frac{i}{F^2}\left\{ \delta^{a_1a_2}\delta^{a_3a_4}\, s + \delta^{a_1a_3}\delta^{a_2a_4}\, t + \delta^{a_1a_4}\delta^{a_2a_3}\, u \right\}$$

$$+ \text{off-shell}$$

This last line <u>is</u> the result for $\pi\pi$ scattering! We just need to replace $p_i \to -p_i$ for the outgoing momenta, i.e. $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$.

Note: If one includes the quark masses, one instead obtains

$$\mathcal{M} = \frac{i}{F^2} \Big\{ \delta^{a_1 a_2} \delta^{a_3 a_4} (s - M_\pi^2) + \delta^{a_1 a_3} \delta^{a_2 a_4} (t - M_\pi^2) + \delta^{a_1 a_4} \delta^{a_2 a_3} (u - M_\pi^2) \Big\}$$

d.) With $g(\vec{\pi}^2) = 1 + \alpha\vec{\pi}^2$

$$U(x) = 1 + i\vec{\pi}\vec{\sigma} - \frac{\vec{\pi}^2}{2} - i\vec{\pi}\cdot\sigma\left(\frac{1}{6}-\alpha\right)\vec{\pi}^2$$

$$+ O(\pi^4)$$

$$\partial_\mu U = i\,\partial_\mu\vec{\pi}\cdot\vec{\sigma} - \vec{\pi}\,\partial_\mu\vec{\pi} - i\,\partial_\mu\vec{\pi}\cdot\vec{\sigma}\,\vec{\pi}^2\left(\frac{1}{6}-\alpha\right)$$

$$- i\,\vec{\pi}\cdot\vec{\sigma}\,2\vec{\pi}\,\partial_\mu\vec{\pi}\left(\frac{1}{6}-\alpha\right)$$

$$\mathcal{L}_{eff} = \frac{F^2}{4}\langle\partial_\mu U(\partial^\mu U)^\dagger\rangle$$

$$= \frac{1}{2}(\partial_\mu\vec{\pi})^2 + \frac{1}{2F^2}(\vec{\pi}\,\partial_\mu\vec{\pi})^2$$

$$- \frac{1}{F^2}\left(\frac{1}{6}-\alpha\right)(\partial_\mu\vec{\pi})^2\,\vec{\pi}^2$$

$$- \frac{2}{F^2}\left(\frac{1}{6}-\alpha\right)(\vec{\pi}\,\partial_\mu\vec{\pi})^2 \qquad (*)$$

Agrees with previous result for $\alpha = 0$. ✓

Consider the integration-by-part identity:

$$\partial_\mu \left((\vec{\pi} \cdot \partial_\mu \vec{\pi}) \vec{\pi}^2 \right)$$

$$\hat{=} (\partial_\mu \vec{\pi})^2 \vec{\pi}^2 + (\vec{\pi} \, \Box \, \vec{\pi}) \vec{\pi}^2$$

$$+ 2 (\vec{\pi} \cdot \partial_\mu \vec{\pi})^2$$

$$\longrightarrow \quad 2 (\vec{\pi} \partial_\mu \vec{\pi}) + (\partial_\mu \vec{\pi})^2 \hat{=} - \vec{\pi}^2 (\vec{\pi} \, \Box \, \vec{\pi})$$

$$\mathcal{L}_{eff} \hat{=} \frac{1}{2} (\partial_\mu \vec{\pi})^2 + \frac{1}{2F^2} (\vec{\pi} \partial_\mu \vec{\pi})^2$$

$$+ \frac{1}{F^2} \left(\frac{1}{6} - \alpha \right) \underline{\vec{\pi}^2 (\vec{\pi} \, \Box \, \vec{\pi})}$$

The Feynman rule for this term

takes the form $\delta^{a_1 a_2}\delta^{a_3 a_4} k_4^2$ + permutations, but $k_i^2 = 0$ on the mass shell.

→ Scattering amplitude is independent of α and agrees with previous result.

Equally well, one could use (*), derive Feynman rules and compute. This would be tedious but lead to the same conclusion. Note that $\alpha = \frac{1}{6}$ gives the simplest form of $\mathcal{L}_{eff}$. This choice corresponds to the σ-parameterization (4.144) in the script.

3a.) The external current part of $\mathcal{L}$ is

$$\Delta\mathcal{L} = l_\mu^i \, \bar{\psi}_L \gamma^\mu t^i \psi_L$$

$$+ r_\mu^i \, \bar{\psi}_R \gamma^\mu t^i \psi_R$$

u(2) matrix!

$$= \bar{\psi}_L \gamma^\mu l_\mu \psi_L + \bar{\psi}_R \gamma^\mu r_\mu \psi_R$$

Perform local chiral transformation $\psi_L \to V_L(x)\psi_L$

$$\mathcal{L}_{kin} = \bar{\psi}_L i \partial\!\!\!/ \psi_L \to \bar{\psi}_L V_L^\dagger i \partial\!\!\!/ V_L \psi_L$$

$$= \bar{\psi}_L i \partial\!\!\!/ \psi_L + \bar{\psi}_L \left(V_L^\dagger i \partial\!\!\!/ V_L \right) \psi$$

$V_L(x)$

The current $\Delta\mathcal{L}$ yields

cancel!

$$\Delta\mathcal{L} \to \Delta\mathcal{L} - i \bar{\psi}_L V_L^\dagger \gamma^\mu (\partial_\mu V_L) \psi_L$$

b.)

$$i D_\mu U = i\partial_\mu U + r_\mu U - U \ell_\mu$$

$$\to \quad (i\partial_\mu V_R) U V_L^\dagger + V_R (i\partial_\mu U) V_L^\dagger + V_R U \partial_\mu V_L^\dagger$$

$$i\partial_\mu V_R U V_L^\dagger + V_R r_\mu \cancel{V_R^\dagger} \cancel{V_R} U V_L^\dagger$$

$$- i(\partial_\mu V_R) \cancel{V_R^\dagger} \cancel{V_R} U V_L^\dagger - V_R U \cancel{V_L^\dagger} \cancel{V_L} \ell_\mu V_L^\dagger$$

$$+ V_R U V_L^\dagger \, i \underbrace{(\partial_\mu V_L) V_L^\dagger}_{-V_L(\partial_\mu V_L^\dagger)}$$

$$= V_R (i D_\mu U) V_L^\dagger$$

$\leadsto$ $\mathrm{tr}\left[(D_\mu U)(D^\mu U)^\dagger\right]$ is invariant.