Exercise Sheet 6

1.) $\gamma_m = 2\alpha_s \dfrac{dZ_{(1)}}{d\alpha_s} = -6C_F \dfrac{\alpha_s}{4\pi}$

$$\longrightarrow \gamma_m^{(0)} = -6C_F$$

$$\mu \frac{dm}{d\mu} = \gamma_m^{(0)} \frac{\alpha_s}{4\pi}$$

$$\longrightarrow \int_{m(\mu_0)}^{m(\mu)} \frac{dm}{m} = \int_{\mu_0}^{\mu} \frac{d\mu}{\mu} \frac{\alpha_s}{4\pi} \gamma_m^{(0)} = \int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{d\alpha}{-2\alpha} \frac{\gamma_0}{\beta_0}$$

$$\ln\left(\frac{m(\mu)}{m(\mu_0)}\right) = -\frac{\gamma_m^{(0)}}{2\beta_0} \ln\left(\frac{\alpha(\mu)}{\alpha(\mu_0)}\right)$$

$$\longrightarrow m(\mu) = m(\mu_0) \left(\frac{\alpha(\mu)}{\alpha(\mu_0)}\right)^{-\frac{\gamma_m^0}{2\beta_0}}$$

Exponent $-\dfrac{\gamma_m}{2\beta_0} > 0$, $\alpha(\mu) > \alpha(\mu_0)$ for $\mu < \mu_0$

$\rightsquigarrow m(\mu) > m(\mu_0)$ for $\mu < \mu_0$.

2.)

a.) $$\frac{d}{d\ln\mu}\alpha_s^{(0)} = 0 = \frac{d}{d\ln\mu} Z_g^2 \mu^{2\varepsilon}\alpha_s(\mu)$$

$$\rightarrow \Big(\underbrace{\frac{d}{d\ln\mu} Z_g^2}_{2 Z_g \frac{d}{d\ln\mu} Z_g}\Big)\mu^{2\varepsilon}\alpha_s(\mu) + 2\varepsilon Z_g^2 \alpha_s(\mu)\mu^{2\varepsilon}$$

$$+ \mu^{2\varepsilon} Z_g^2 \frac{d\alpha_s}{d\ln\mu} = 0$$

$$\rightarrow \frac{d\alpha_s}{d\ln\mu} = -2\varepsilon\alpha_s(\mu) - 2\alpha_s Z_g^{-1}\frac{dZ_g}{d\ln\mu} = \beta(\alpha_s,\varepsilon) \quad (*)$$

b.)

$$(*)\cdot Z_g = \beta(\alpha_s,\varepsilon) Z_g = -2\varepsilon Z_g \alpha_s$$

Poles only!

$$-2\alpha_s \frac{\partial Z_g}{\partial\alpha_s}\beta(\alpha_s,\varepsilon) \quad (**)$$

Expand $\beta(\alpha_s,\varepsilon) = \beta(\alpha_s) + \sum_{n=1}^{\infty}\varepsilon^n \beta_{[n]}(\alpha_s)$

Take coefficient of ε^n-term in $(**)$.

RHS only has terms up to $O(\varepsilon)$!

$\longrightarrow$ Higher order terms in ε must vanish on LHS

$\longrightarrow$ $\beta_{[n]} = 0$ for $n > 1$.

The $O(\varepsilon)$-terms are:

$$\beta_{[1]} \cdot 1 \cdot \varepsilon = -2 \varepsilon \alpha_s$$

The $O(\varepsilon^0)$-terms are

$$\beta(\alpha_s) + \cancel{\overbrace{\beta_{[1]}(\alpha_s)}^{-2\alpha_s} \cdot Z_{g[1]}} = \cancel{-2\alpha_s Z_{g[1]}}$$

$$- 2\alpha_s \frac{\partial Z_{g[1]}}{\partial \alpha_s} (-2\alpha_s)$$

$$\beta(\alpha_s) = 4\alpha_s^2 \frac{\partial Z_{g[1]}}{\partial \alpha_s} .$$

Now consider bare Wilson coefficient, e.g. bare mass from exercise 1.

$$\frac{d}{d\ln\mu} m_q = 0 = \frac{d}{d\ln\mu}\left(Z_m m(\mu) \right)$$

$$\to \left(\frac{d}{d\ln\mu} Z_m\right) m(\mu) + Z_m \underbrace{\frac{d}{d\ln\mu} m(\mu)}_{\gamma_m(\mu,\varepsilon)\, m(\mu)} = 0$$

pure pole

$$\to \left(\frac{d}{d\alpha_s} Z_m\right) \underbrace{\beta(\alpha_s,\varepsilon)}_{-2\alpha_s\varepsilon + \beta(\alpha_s)\ \text{(previous result.)}} + Z_m \gamma_m(\mu,\varepsilon) = 0 \qquad (***)$$

$$\gamma_m(\mu,\varepsilon) = \gamma_m + \sum_{n=1}^{\infty} \varepsilon^n \gamma_{[n]}$$

We immediately see that $\gamma_{[n]} = 0$ for $n \geq 1$.

For the ε^0 - term, we get:

$$\frac{d}{d\alpha_s} Z_{m[1]} (-2\alpha_s) + \gamma_m = 0$$

$$\longrightarrow \quad \gamma_m = 2\alpha_s \frac{dZ_{m[1]}}{d\alpha_s}$$ "magic relation"