Solution Exercise 5

1.)

a.) $U^+ = \exp(-i (T^a)^+ \alpha^a)$

$U^{-1} = \exp(-i \, T^a \alpha^a)$

Unitarity : $U^+ = U^{-1} \rightarrow (T^a)^+ = T^a$

$\det e^A = e^{\mathrm{tr}(A)}$

$\det U \overset{!}{=} 1 \rightarrow \mathrm{tr}(T^a \alpha^a) = 0 \quad \forall \alpha^a$

$\rightarrow \mathrm{tr}(T^a) = 0$

b.) $[[T^a, T^b], T^c] = i f^{abd} [T^d, T^c]$

$= - f^{abd} f^{dce} \, T^e$

Rewriting all three terms in the commutator Jacobi identity directly gives the one of the structure constants times $-T^e$.

To show that f^{abc} are real, take the hermitean conjugate of (1)

$$[(T^b)^+, (T^c)^+] = -i(f^{abc})^* (T^c)^+$$

but from (1) we know that $(T^a)^+ = T^a$

$$\rightarrow [T^b, T^c] = -i(f^{abc})^* T^c$$

$$\overset{(1)}{=}$$

$$-i f^{abc} T^c$$

$$\rightarrow (f^{abc})^* = f^{abc}.$$

c.) $[T_A^a, T_A^e]_{bc} = (T_A^a)_{bd} (T^e)_{dc}$

$$- (T^e)_{bd} (T^a)_{dc}$$

$$= -f^{abd} f^{edc} + f^{ebd} f^{adc}$$

$$= -f^{ebd} f^{dce} - f^{cad} f^{dbe}$$

$$= f^{bcd} f^{dae} = i f^{aed} \left(-i f^{dbc}\right)$$

↑
Jacobi

$$= i f^{aed} (T^d)_{bc} \quad ✓$$

2.)

Take the transpose of (1)

$$\left[(T^b)^T, (T^a)^T \right] = i f^{abc} (T^c)^T$$

$$\rightarrow \left[(-T^a)^T, (-T^b)^T \right] = i f^{abc} (-T^c)^T$$

→ For any representation, also $(-T^c)^T$ is one.

For the adjoint representation the two are equal:

$$(T^a_{\bar{A}})_{bc} = -(T^a_A)^*_{bc} = -(+i f^{abc}) = -i f^{abc} = (T^a_A)_{bc}$$

3.)

summation over a

a.) $[C_R, T_R^b] = [T_R^a T_R^a, T_R^b]$

$= T_R^a [T_R^a, T_R^b] + [T_R^a, T_R^b] T_R^a$

$= i \underbrace{f^{abc}}_{\text{anti-symm}} \underbrace{(T_R^a T_R^c + T_R^c T_R^a)}_{\text{symm}} = 0$

b.) $t^a \cdot t^a = C_F \cdot \mathbb{1}$

$$\mathrm{tr}(t^a t^a) = C_F \cdot N_c$$

$$\|$$

$$T_F \, \delta^{aa} = T_F (N_c^2 - 1)$$

$$\rightarrow C_F = T_F \frac{N_c^2 - 1}{N_c} = \frac{N_c^2 - 1}{2N_c} .$$

For C_A:

$$\mathrm{tr}\left(T_A^a \; T_A^b \right) = \left(-i f^{acd} \right)\left(-i f^{bdc} \right)$$

$$= f^{acd} \, f^{bcd} \, .$$

$$\mathrm{tr}\left([t^a, t^c][t^b, t^d] \right) = \left(-i f^{ace} \right)\left(-i f^{bdg} \right) \cdot \underbrace{\mathrm{tr}\left(t^e t^g \right)}_{T_F \cdot \delta^{eg}}$$

$$= -\frac{1}{2} f^{ace} f^{bde}$$

$$\to \mathrm{tr}\left(T_A^a T_A^a \right) = C_A \, \mathrm{tr}\left(\mathbb{1}_A \right)$$

$$= C_A \left(N_c^2 - 1 \right)$$

$$= f^{acd} f^{acd}$$

$$= -2 \, \mathrm{tr}\left([t^a, t^c][t^a, t^c] \right)$$

$$= -2\,\mathrm{tr}\Big(t^a t^c t^a t^c - t^c \overbrace{t^a t^a}^{C_F} t^c$$

$$- t^a \underbrace{t^c t^c}_{C_F} t^a + t^c t^a t^c t^a \Big)$$

$$= -4\,\mathrm{tr}\Big(t^a t^c t^a t^c - C_F^2 \cdot \mathbb{1} \Big)$$

$$\left[\ \mathrm{tr}\left(t^a t^c t^a t^c \right) \underset{\text{Fierz}}{=} \frac{1}{2}\mathrm{tr}\,\overbrace{\left(t^c \right)}^{=0}\mathrm{tr}\left(t^c \right) - \frac{1}{2N_c}\mathrm{tr}\,(\underbrace{t^c t^c}_{C_F \cdot \mathbb{1}}) \right.$$

$$= +\frac{2}{N_c} C_F \cdot N_c + 4 C_F^2 \cdot N_c$$

$$= N_c (N_c^2 - 1) \quad \Rightarrow \quad C_A = N_c$$

Bonus exercise:

Any hermitian matrix can be written

as $M = c_0 \mathbb{1} + c_a t^a$

Note: $\mathrm{tr}[t^b M] = c_a \mathrm{tr}(t^b t^a)$

$= T_F c^b = \frac{1}{2} c^b$

$\mathrm{tr}[\mathbb{1} M] = c_0 \mathrm{tr}(\mathbb{1}) = c_0 \cdot N$

$$\rightarrow \boxed{M = \frac{1}{N} \mathrm{tr}[M \mathbb{1}] \mathbb{1} + 2 \cdot \mathrm{tr}[M t^a] t^a}$$

In components:

$$M_{ij} = \frac{1}{N} M_{ek} \delta_{ke} \delta_{ij}$$

$$+ 2 M_{ek} t^a_{ke} t^a_{ij} = M_{ek} \delta_{ie} \delta_{jk}$$

This must hold for any $M_{\ell k}$

$$\to \quad t^a_{ij}\, t^a_{k\ell} = -\frac{1}{2N}\, \delta_{ij}\, \delta_{k\ell} + \frac{1}{2}\, \delta_{i\ell}\, \delta_{jk}$$