3. Continuum effective theory

The construction of the Wilsonian effective action is very physically intuitive, and leads to a new picture of renormalization. However, actually integrating out the physics above a cut-off Λ is often as difficult as solving the theory.

Instead of actually integrating out the high-energy physics, it is much simpler to work without a hard cut-off and to treat the effective theory like a standard continuum field theory.

To get the effective theory, one follows a number of steps, which we now discuss in turn.

1.) <u>Identify the low-energy degrees of freedom.</u>

This <u>can</u> be simple: e.g. if we consider a theory with a very heavy particle and weak coupling, the low-energy degrees of freedom are simply all light particles.

In other cases it's non-trivial: In QCD, the low-energy degrees of freedom are π's, K's, p, n, etc. and not the quarks and gluons in the high-energy Lagrangian.

2.) Construct the most general low-energy $\mathcal{L}_{eff}$ consistent with the symmetries of the full theory. Order the operators in $\mathcal{L}_{eff}$ by their dimension.

3.) Matching. To determine the coupling constants in $\mathcal{L}_{eff}$ calculate a number of correlation functions (or scattering amplitudes) in both the full and the effective theory. Expand the full theory result around the low-E limit and adjust the Wilson coefficients in $\mathcal{L}_{eff}$ such that the full and EFT results agree.

4.) RG improvement. The perturbative expansion of the Wilson coefficients can be improved by solving RG equations for the coefficients.

It is simplest to use dimensional regularization (and the $\overline{MS}$ scheme) in both the full and the effective theory. At first sight it seems troubling to work without a hard cut-off and to integrate up to arbitrarily high energies even in the low-E theory, which is not valid at high energies: However, we know from Wilson that we can absorb arbitrary high-E physics into the couplings of $\mathcal{L}_{eff}$. By adjusting the couplings, we can thus obtain the correct low-E results despite the incorrect behavior of our amplitudes at high energies.

Let us use a toy model with a heavy and a light scalar field to illustrate the above steps. Our full theory is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_L)^2 - \frac{m^2}{2}\phi_L^2 + \frac{1}{2}(\partial_\mu \phi_H)^2 - \frac{M^2}{2}\phi_H^2$$

$$- \frac{\lambda_L}{4!}\phi_L^4 - \frac{\lambda_H}{4!}\phi_H^4 - \frac{\lambda_{HL}}{2!2!}\phi_L^2\phi_H^2$$

$$- \frac{g}{2!}\phi_H\phi_L^2$$

Note that $\mathcal{L}$ is symmetric under $\phi_L \to -\phi_L$. To renormalize the theory we need to include also

$$\delta\mathcal{L} = A + B\phi_H + C\phi_H^3$$

but we assume that A, B, C are renormalized to zero.

Now let's follow the different steps to construct $\mathcal{L}_{eff}$.

1.) Low-E degrees of freedom: ϕ_L

2.) Effective Lagrangian

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial_\mu \phi_L)^2 - \frac{\tilde{m}^2}{2}\phi_L^2 - \frac{\tilde{\lambda}}{4!}\phi_L^4 - \frac{1}{2}\frac{C_{2,4}}{M^2}\phi\Box^2\phi$$

$$-\frac{1}{6!}\frac{C_{6,0}}{M^2}\phi_L^6 - \frac{1}{4!}\frac{C_{4,2}}{M^2}\phi^2\Box\phi^2$$

$$+ O\left(\frac{1}{M^4}\right)$$

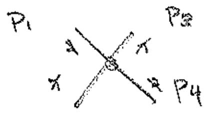
Exercise: Show that all other $d=6$ operators reduce to these three after integration by part.

3.) Matching. To extract the values of $\tilde{m}$, $\tilde{\lambda}$, c_6 and $C_{4,2}$ let's calculate the 2-, 4- and 6-point functions.

3.1 Tree-level matching

Effective theory computation:

$$i\Gamma_2 = iG^{-1}(p) = (p^2 - \tilde{m}^2) \times (-1)$$

$i\Gamma_4 =$ 

$+$ 

$\tilde{\lambda}$ $\qquad\qquad$ $C_{4,2}$

$$= \tilde{\lambda} - \frac{C_{4,2}}{M^2}\frac{1}{3}\left[(p_1+p_2)^2 + (p_1-p_3)^2 + (p_1-p_4)^2\right]$$

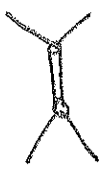
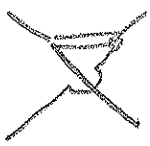
(see Appendix B)

$i\Gamma_6 = \dfrac{C_6}{M^2} +$  $+ \ldots$

Full theory computation

$$C_{2,4} = 0 + O(\lambda)$$

$$i\Gamma_2 = iG^{-1}(p) = p^2 - m^2 \quad \Longrightarrow \quad \underline{\tilde{m} = m + O(\lambda)}$$

$i\Gamma_4 =$  $+$  $+$  $+$ 

λ_L $\qquad\qquad$ $g \quad g$

$$= \lambda_L + i(ig)^2\left[\frac{i}{(p_1+p_2)^2 - M^2} + \frac{i}{(p_1-p_3)^2 - M^2} + \frac{i}{(p_1-p_4)^2 - M^2}\right]$$

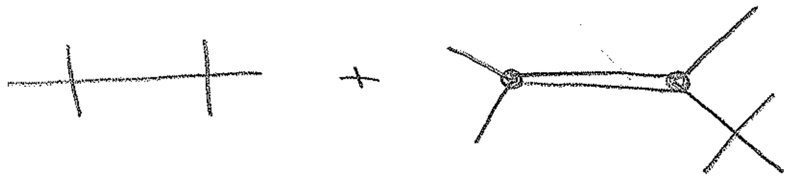
Since $p_i^\mu \ll M$ at low energies, we can Taylor expand Γ_4

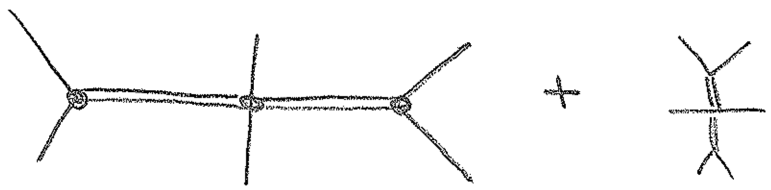
$$= i\Gamma_4 = \lambda_L - \frac{3g^2}{M^2} - \frac{g^2}{M^4}\left[(p_1+p_2)^2 + (p_1-p_3)^2 + (p_1-p_4)^2\right]$$

Comparison with the EFT result gives

$$\underline{\tilde{\lambda} = \lambda_L - \frac{3g^2}{M^2}}$$

$$\underline{c_{(4,2)} = +\frac{3g^2}{M^2}}$$

$i\Gamma_6 =$  $+ \ldots$

$+$  $+ \ldots$

The diagrams on the first line are one-particle reducible with respect to the light field ϕ_L.

These diagrams are automatically reproduced since we matched the 4-point function.

Only the diagrams on the second line will contribute to the matching on C_6. Since the operator does not involve derivatives it is sufficient to compute Γ_6 for vanishing momenta. The matching condition is

$$\frac{C_6}{M^2} = i(-ig)^2(-i\lambda_{HL}) \cdot \left(\frac{i}{-M^2}\right)^2 \cdot 45$$

$$C_6 = +45\,\lambda_{HL}\,\frac{g^2}{M^2}\,.$$

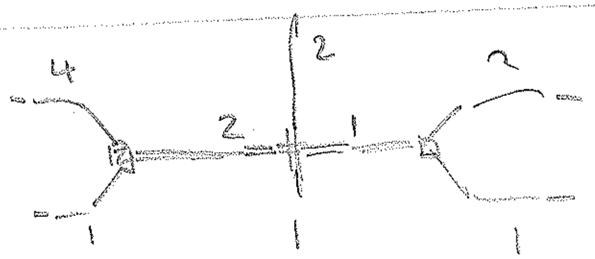
prefactors of $\phi_L^2\phi_H$ and $\phi_H^2\phi_L^2$

$$45 = \underbrace{6!}_{\text{perms}} \left(\frac{1}{2}\right)^2 \frac{1}{4}$$

This completes the construction of the effective theory at the tree level.

Appendix: Combinatorics for the 6-point function

$$\frac{1}{2}\left(\frac{1}{2!}\right)^2 \frac{1}{2!2!}$$



$$= 4 \cdot 2 \cdot 2 \cdot 2$$

Each of the diagrams has a factor 1.

How many distinct ones are there?

Number of pairs (contractions?)

$$\binom{6}{2} = \frac{6!}{2!\,(6-2)!} = 15$$

once a pair has been picked, four lines remain and one chooses again a pair

$$\binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

but there is a symmetry, see the above diagram, so one has to divide by 2: $15 \cdot 6/2 = \underline{45}$