EFT Series 5

Exercise 1

The Gordon Identity reads

$$\bar{u}(p_2)\gamma^\mu u(p_1) = \frac{1}{2m}\bar{u}(p_2)[(p_1 + p_2)^\mu + i\sigma^{\mu\nu} q_\nu]u(p_1),$$

where $q = p_2 - p_1$.

It can be derived using the following relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta_{\mu\nu} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

$$(p_1 + p_2)^\mu + i\sigma^{\mu\nu} q_\nu = p_1^\mu + p_2^\mu - \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p_{2\nu} - p_{1\nu})$$

$$= p_1^\mu + p_2^\mu - \frac{1}{2}(\gamma^\mu\gamma^\nu p_{2\nu} - \gamma^\mu\gamma^\nu p_{1\nu} - \gamma^\nu\gamma^\mu p_{2\nu} + \gamma^\nu\gamma^\mu p_{1\nu})$$

$$= p_1^\mu + p_2^\mu - \frac{1}{2}(p_{2\nu}[2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu] - \gamma^\mu \not{p}_1 - \not{p}_2\gamma^\mu + [2\eta^{\mu\nu} - \gamma^\mu\gamma^\nu]p_{1\nu})$$

$$= p_1^\mu + p_2^\mu - p_{2\nu}\eta^{\mu\nu} + \frac{1}{2}\not{p}_2\gamma^\mu + \frac{1}{2}\gamma^\mu\not{p}_1 + \frac{1}{2}\not{p}_2\gamma^\mu - \eta^{\mu\nu}p_{1\nu} + \frac{1}{2}\gamma^\mu\not{p}_{1\nu}$$

$$= p_1^\mu + p_2^\mu - \eta^{\mu\nu}p_{2\nu} + \not{p}_2\gamma^\mu + \gamma^\mu\not{p}_1 - \eta^{\mu\nu}p_{1\nu}$$

This can be simplified using the equation of motion:

$$\not{p}u(p) = mu(p); \qquad \bar{u}(p)\not{p} = \bar{u}(p)m \tag{1}$$

$$\bar{u}(p_2)[(p_1+p_2)^\mu+i\sigma^{\mu\nu}q_\nu]u(p_1) = \bar{u}(p_2)[p_1^\mu+p_2^\mu-\eta^{\mu\nu}p_{2\nu}-\eta^{\mu\nu}p_{1\nu}+\not{p}_2\gamma^\mu+\gamma^\mu\not{p}_1]u(p_1)$$

$$= \bar{u}(p_2)[\eta^{\mu\nu} p_{1\nu} + \eta^{\mu\nu} p_{2\nu} - \eta^{\mu\nu} p_{2\nu} - \eta^{\mu\nu} p_{1\nu} + 2m\gamma^\mu] u(p_1)$$

So

$$\frac{1}{2m}\bar{u}(p_2)[(p_1+p_2)^\mu + i\sigma^{\mu\nu} q_\nu] u(p_1) = \frac{1}{2m}\bar{u}(p_2)[2m\gamma^\mu] u(p_1) = \bar{u}(p_2)\gamma^\mu u(p_1).$$

Exercise 2

We want to compute the causal propagator $i\Delta(x-y) = \langle 0| \left[\phi(x), \phi^\dagger(y)\right] |0\rangle$ for a free scalar field for a particle of mass μ using

$$\phi(x) = \int \frac{d^3p}{\sqrt{2E_p}(2\pi)^{3/2}} [a_p e^{-ipx} + a_p^\dagger e^{ipx}]$$

$$\left[a_p, a_{p'}^\dagger\right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p'}); \quad [a_p, a_{p'}] = 0;$$

$$\left[\phi(x), \phi^\dagger(y)\right] = \left[\int \frac{d^3p}{\sqrt{2E_p}(2\pi)^{3/2}}(a_p e^{-ipx} + a_p^\dagger e^{ipx}), \int \frac{d^3q}{\sqrt{2E_q}(2\pi)^{3/2}}(a_q e^{-iqy} + a_q^\dagger e^{iqy})\right]$$

$$\left[\phi(x), \phi^\dagger(y)\right] = \int \frac{d^3p}{\sqrt{2E_p}(2\pi)^{3/2}} \int \frac{d^3q}{\sqrt{2E_q}(2\pi)^{3/2}} (\left[a_p, a_q^\dagger\right] e^{i(-px+qy)} + \left[a_p^\dagger, a_q\right] e^{i(px-qy)})$$

$$\left[\phi(x), \phi^\dagger(y)\right] = \int \frac{d^3p}{\sqrt{2E_p}(2\pi)^{3/2}} \int \frac{d^3q}{\sqrt{2E_q}(2\pi)^{3/2}} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})(e^{-i(px-qy)} - e^{i(px-qy)})$$

Now use the relation $\int d^3q \delta^{(3)}(\vec{p} - \vec{q}) f(\vec{q}) = f(\vec{p})$,

$$\left[\phi(x), \phi^\dagger(y)\right] = \int \frac{d^3p}{2E_p}(e^{-ip(x-y)} - e^{ip(x-y)}).$$

Finally use

$$\int \frac{d^3p}{2E_p} = \int d^4p \delta(p^2 - \mu^2)\theta(p^0).$$

So

$$\langle 0| \left[\phi(x), \phi^\dagger(y)\right] |0\rangle = \langle 0| \int d^4p\delta(p^2-\mu^2)\theta(p^0)(e^{-ip(x-y)} - e^{ip(x-y)})|0\rangle$$

and

$$i\Delta(x-y) = \underbrace{\langle 0|0\rangle}_{=1} \int d^4p\delta(p^2-\mu^2)\theta(p^0)(e^{-ip(x-y)} - e^{ip(x-y)})$$

If x and y are spacelike separated, one can find a frame (of equal times) where $x-y=(0,\vec{z})$. Then

$$i\Delta(x-y) \propto \int d^4p(e^{-ip(x-y)} - e^{ip(x-y)}) = 0,$$

as the integral is taken over an odd function. This means that the contributions to the propagator $[\phi(x), \phi^\dagger(y)]$ cancel outside the light-cone and causality is preserved.

Exercise 3

We want to show that

$$\langle 0|[\phi(x), \phi^\dagger(y)]|0\rangle = i\int d\mu^2\rho(\mu^2)\Delta(x-y), \tag{2}$$

$$\langle 0|[\phi(x), \phi^\dagger(y)]|0\rangle = \int d^4p \int d\mu^2\delta(p^2-\mu^2)\rho(\mu^2)\theta(p^0)(e^{-ip(x-y)} - e^{ip(x-y)}). \tag{3}$$

Insert a complete set of eigenstates,

$$\langle 0|[\phi(x), \phi^\dagger(y)]|0\rangle = \sum_x \left[\langle 0|\phi(x)|x\rangle\langle x|\phi^\dagger(y)|0\rangle - \langle 0|\phi^\dagger(y)|x\rangle\langle x|\phi(x)|0\rangle\right] \tag{4}$$

Use the translation operator $\boldsymbol{P}x$, $\phi(x) = e^{i\boldsymbol{P}x}\phi(0)e^{-i\boldsymbol{P}x}$. Acting on the vacuum state $|0\rangle$, the operator produces eigenvalue 0, while $\langle 0|\phi(x)|x\rangle = \langle 0|e^{i\boldsymbol{P}x}\phi(0)e^{-i\boldsymbol{P}x}|x\rangle = e^{-ip_x x}\langle 0|\phi(0)|x\rangle$.

$$= \sum_x \left[\langle 0|e^{i\boldsymbol{P}x}\phi(0)e^{-i\boldsymbol{P}x}|x\rangle\langle x|e^{i\boldsymbol{P}y}\phi^\dagger(0)e^{-i\boldsymbol{P}y}|0\rangle - \langle 0|e^{i\boldsymbol{P}y}\phi^\dagger(0)e^{-i\boldsymbol{P}y}|x\rangle\langle x|e^{i\boldsymbol{P}x}\phi(0)e^{-i\boldsymbol{P}x}|0\rangle\right],$$

$$= \sum_x \left[e^{-ip_x(x-y)} \langle 0|\phi(0)|x\rangle\langle x|\phi^\dagger(0)|0\rangle - e^{ip_x(x-y)} \langle 0|\phi^\dagger(0)|x\rangle\langle x|\phi(0)|0\rangle \right].$$

Now insert the relation $p_x = \int d^4p\delta^{(4)}(p - p_x)$,

$$= \int d^4p \sum_x \delta^{(4)}(p-p_x) \left[e^{-ip_x(x-y)} \langle 0|\phi(0)|x\rangle\langle x|\phi^\dagger(0)|0\rangle - e^{ip_x(x-y)} \langle 0|\phi^\dagger(0)|x\rangle\langle x|\phi(0)|0\rangle \right].$$

At this point we define

$$\sum_x \delta^{(4)}(p - p_x)\langle 0|\phi(0)|x\rangle\langle x|\phi^\dagger(0)|0\rangle = \theta(p_0)\rho(p^2) \tag{5}$$

and

$$\sum_x \delta^{(4)}(p - p_x)\langle 0|\phi^\dagger(0)|x\rangle\langle x|\phi(0)|0\rangle = \theta(p_0)\tilde{\rho}(p^2), \tag{6}$$

so that

$$\langle 0|[\phi(x), \phi^\dagger(y)]|0\rangle = \int d^4p \left(e^{-ip(x-y)}\theta(p_0)\rho(p^2) - e^{ip(x-y)}\theta(p_0)\tilde{\rho}(p^2) \right). \tag{7}$$

The commutator must vanish for spacelike four-vectors $x - y = z = (0, \vec{z})$. Replacing $\vec{z}$ by $-\vec{z}$ will not alter the result as we are integrating over all momenta. We perform the replacement in the second term of Eq. (7) and find

$$\langle 0|[\phi(x), \phi^\dagger(y)]|0\rangle = \int d^4p \left(e^{-ipz}\theta(p_0)\rho(p^2) - e^{-ipz}\theta(p_0)\tilde{\rho}(p^2) \right) \overbrace{=}^{!} 0. \tag{8}$$

Equation (8) can be fulfilled if $\rho(p^2) = \tilde{\rho}(p^2)$. Then

$$\langle 0|[\phi(x), \phi^\dagger(y)]|0\rangle = \int d^4p\rho(p^2)\theta(p_0) \left(e^{-ip(x-y)} - e^{ip(x-y)} \right).$$

Now we insert $\int d\mu^2\delta(p^2 - \mu^2) = 1$,

$$\langle 0|[\phi(x), \phi^\dagger(y)]|0\rangle = \int d^4p \int d\mu^2\delta(p^2 - \mu^2)\rho(p^2)\theta(p_0) \left(e^{-ip(x-y)} - e^{ip(x-y)} \right),$$

as was to be shown.

Monika Hager, HS 2015