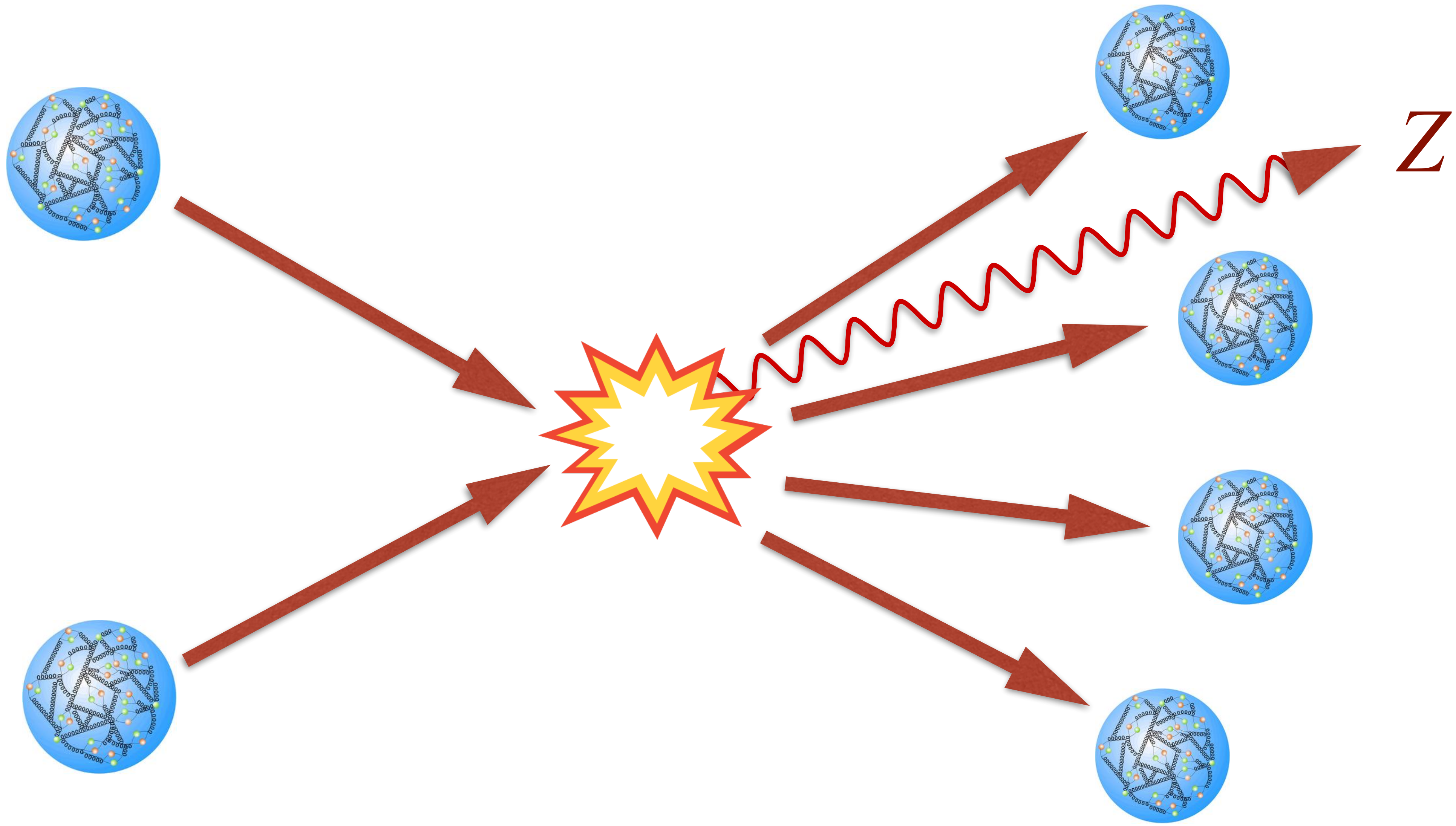
QCD and Monte Carlo

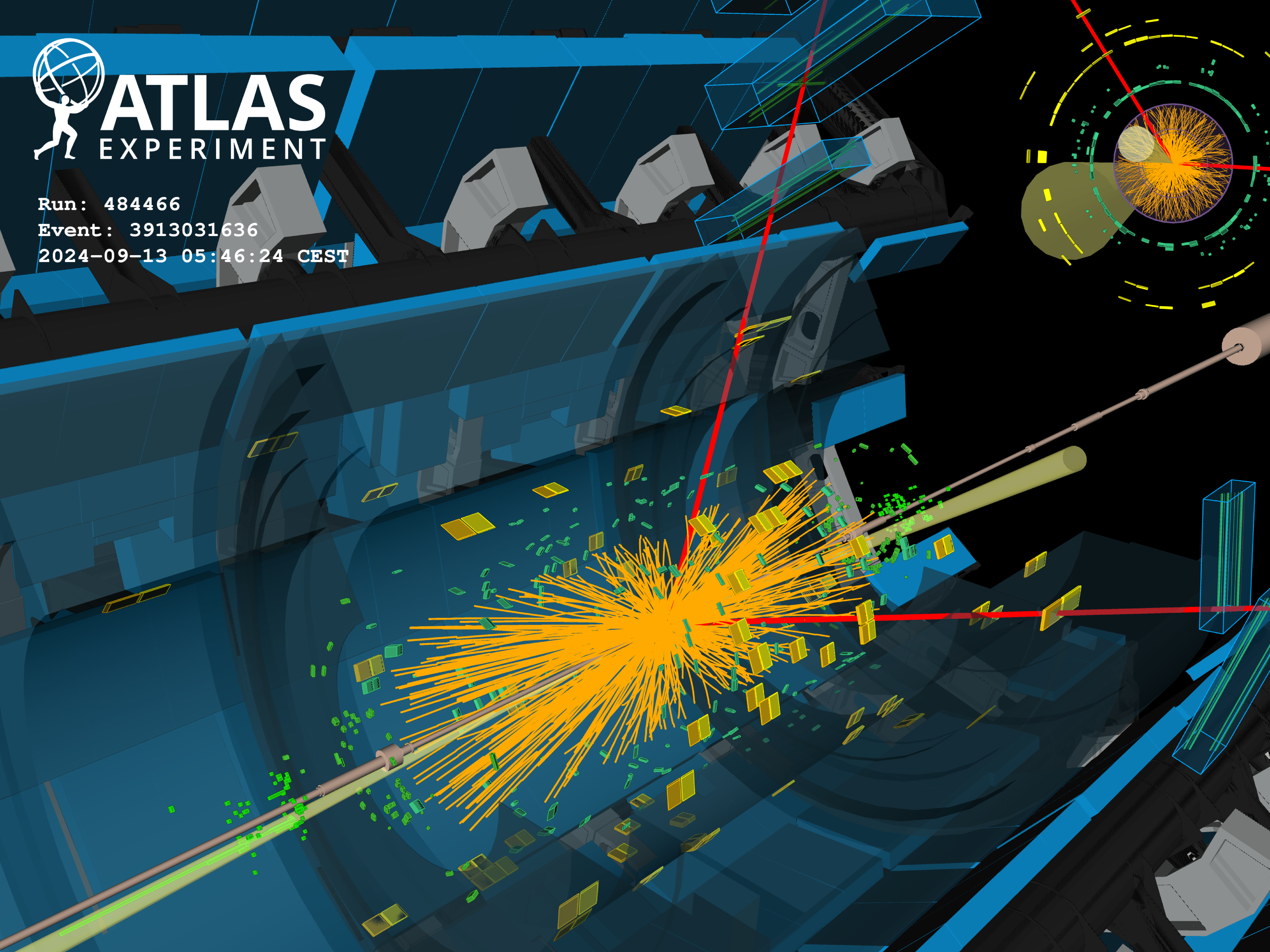
Thomas Becher

University of Bern

CERN-Fermilab Hadron Collider Physics Summer School 2025



The LHC collides very energetic hadrons, complicated relativistic bound states of quarks and gluons, which scatter into a huge number of hadrons + EW particles.


ATLAS
EXPERIMENT
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Event: 3913031636
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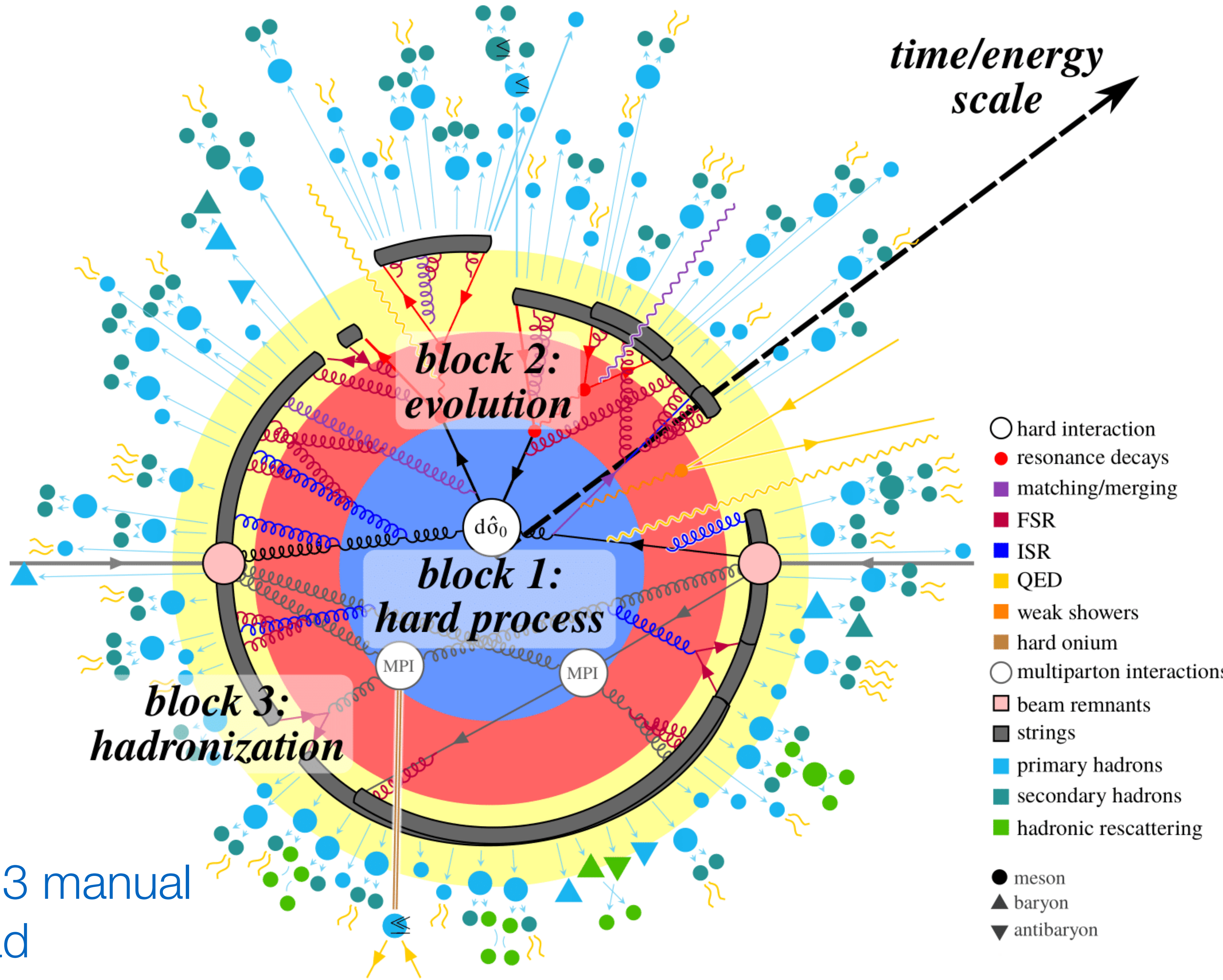
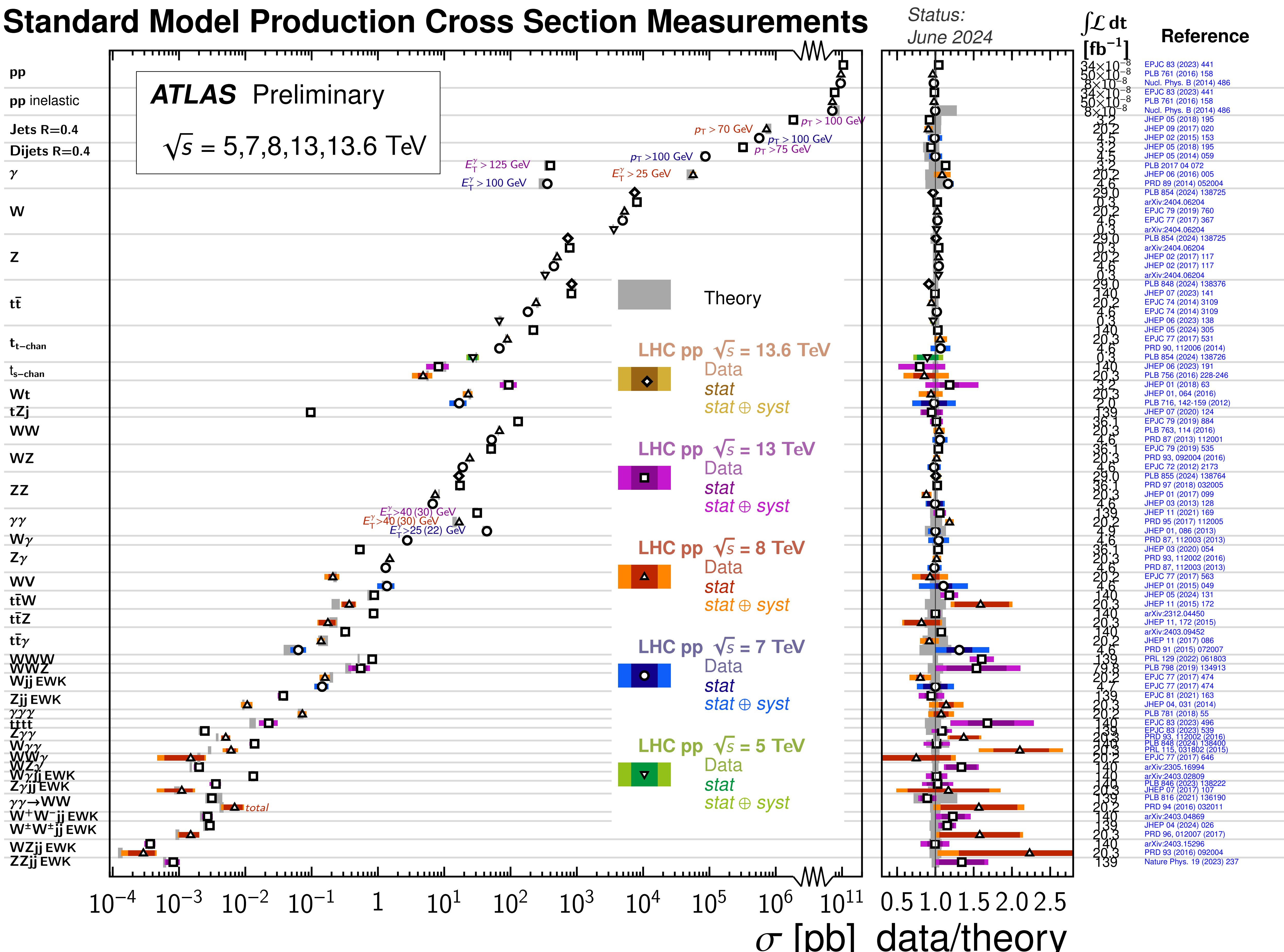
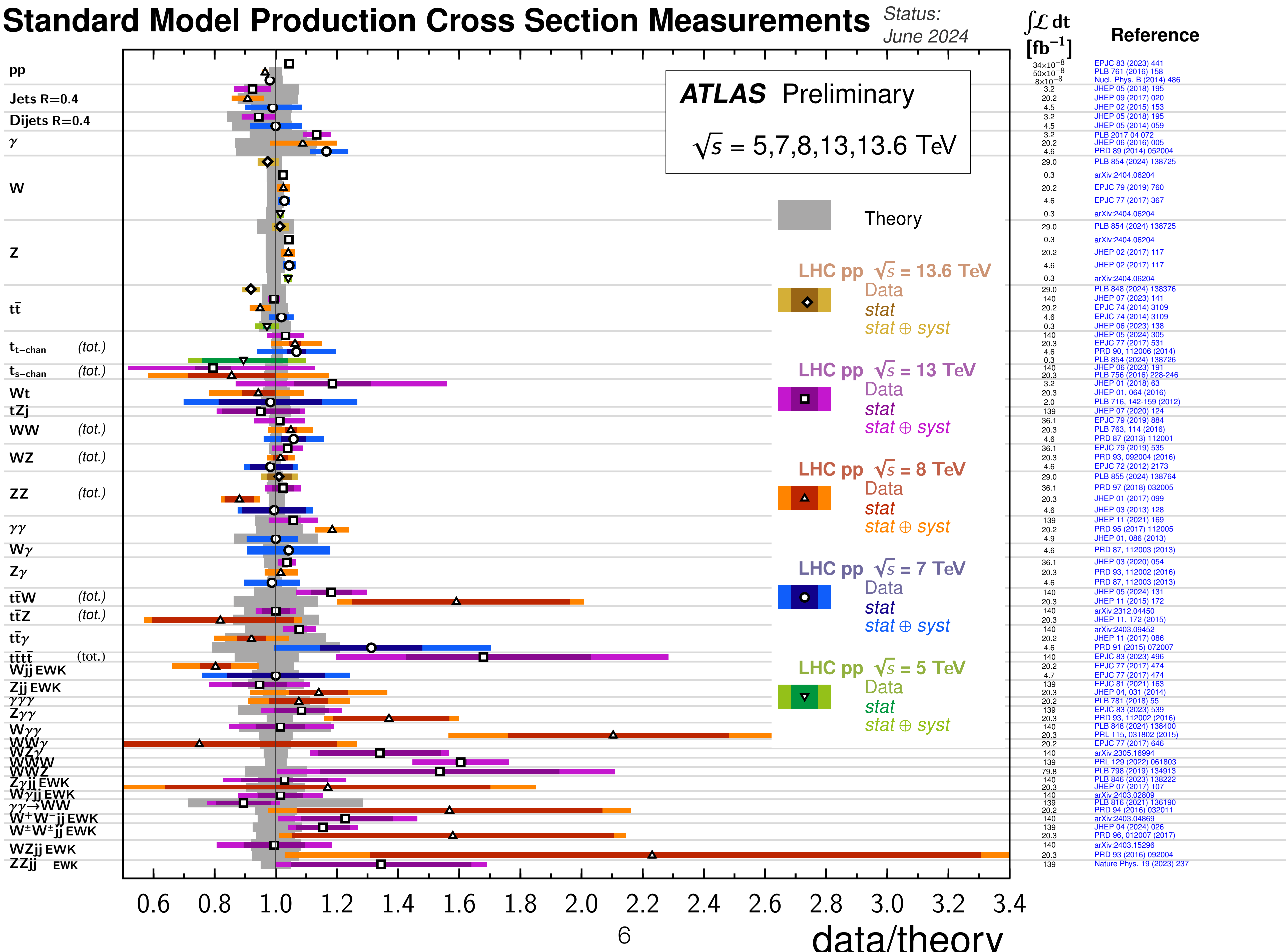


from Pythia 8.3 manual
MLHad

Theoretical predictions are obviously very challenging, mainly due to QCD (strong interaction) effects!



Standard Model Production Cross Section Measurements
Status: June 2024
ATLAS Preliminary
$\sqrt{s}$ = 5,7,8,13,13.6 TeV
$\int \mathcal{L}$ dt [fb^{-1}]
Reference
pp
pp inelastic
Jets R=0.4
Dijets R=0.4
γ
W
Z
tt̄
t$_{t-chan}$
t$_{s-chan}$
Wt
tZj
WW
WZ
ZZ
γγ
Wγ
Zγ
WV
tt̄W
tt̄Z
tt̄γ
WWW
WWZ
Wjj EWK
Zjj EWK
γγγ
tttt
Zγγ
Wγγ
WWγ
WZγ
Wγjj EWK
Zγjj EWK
γγ→WW
W$^+$W$^-$jj EWK
W$^\pm$W$^\pm$jj EWK
WZjj EWK
ZZjj EWK
Theory
LHC pp $\sqrt{s}$ = 13.6 TeV
Data
stat
stat ⊕ syst
LHC pp $\sqrt{s}$ = 13 TeV
Data
stat
stat ⊕ syst
LHC pp $\sqrt{s}$ = 8 TeV
Data
stat
stat ⊕ syst
LHC pp $\sqrt{s}$ = 7 TeV
Data
stat
stat ⊕ syst
LHC pp $\sqrt{s}$ = 5 TeV
Data
stat
stat ⊕ syst
p_T > 70 GeV
p_T > 100 GeV
p_T > 100 GeV
p_T >75 GeV
p_T >100 GeV
E_T^γ > 125 GeV
E_T^γ > 100 GeV
E_T^γ > 25 GeV
E_T^γ>40 (30) GeV
E_T^γ>40 (30) GeV
E_T^γ>25 (22) GeV
total
10^{-4} 10^{-3} 10^{-2} 10^{-1} 1 10^1 10^2 10^3 10^4 10^5 10^6 10^{11}
σ [pb]
0.5 1.0 1.5 2.0 2.5
data/theory



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Data
stat
stat ⊕ syst
LHC pp √s = 7 TeV
Data
stat
stat ⊕ syst
LHC pp √s = 5 TeV
Data
stat
stat ⊕ syst
pp
Jets R=0.4
Dijets R=0.4
γ
W
Z
tt̄
t t−chan (tot.)
t s−chan (tot.)
Wt
tZj
WW (tot.)
WZ (tot.)
ZZ (tot.)
γγ
Wγ
Zγ
tt̄W (tot.)
tt̄Z (tot.)
tt̄γ
tt̄tt̄ (tot.)
Wjj EWK
Zjj EWK
γγγ
Zγγ
Wγγ
WWγ
WZγ
WWW
WWZ
Zγjj EWK
Wγjj EWK
γγ→WW
W+W−jj EWK
W±W±jj EWK
WZjj EWK
ZZjj EWK
0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0 2.2 2.4 2.6 2.8 3.0 3.2 3.4
data/theory
∫L dt [fb−1]
Reference
34×10−8 EPJC 83 (2023) 441
50×10−8 PLB 761 (2016) 158
8×10−8 Nucl. Phys. B (2014) 486
3.2 JHEP 05 (2018) 195
20.2 JHEP 09 (2017) 020
4.5 JHEP 02 (2015) 153
3.2 JHEP 05 (2018) 195
4.5 JHEP 05 (2014) 059
3.2 PLB 2017 04 072
20.2 JHEP 06 (2016) 005
4.6 PRD 89 (2014) 052004
29.0 PLB 854 (2024) 138725
0.3 arXiv:2404.06204
20.2 EPJC 79 (2019) 760
4.6 EPJC 77 (2017) 367
0.3 arXiv:2404.06204
29.0 PLB 854 (2024) 138725
0.3 arXiv:2404.06204
20.2 JHEP 02 (2017) 117
4.6 JHEP 02 (2017) 117
0.3 arXiv:2404.06204
29.0 PLB 848 (2024) 138376
140 JHEP 07 (2023) 141
20.2 EPJC 74 (2014) 3109
4.6 EPJC 74 (2014) 3109
0.3 JHEP 06 (2023) 138
140 JHEP 05 (2024) 305
20.3 EPJC 77 (2017) 531
4.6 PRD 90, 112006 (2014)
0.3 PLB 854 (2024) 138726
140 JHEP 06 (2023) 191
20.3 PLB 756 (2016) 228-246
3.2 JHEP 01 (2018) 63
20.3 JHEP 01, 064 (2016)
2.0 PLB 716, 142-159 (2012)
139 JHEP 07 (2020) 124
36.1 EPJC 79 (2019) 884
20.3 PLB 763, 114 (2016)
4.6 PRD 87 (2013) 112001
36.1 EPJC 79 (2019) 535
20.3 PRD 93, 092004 (2016)
4.6 EPJC 72 (2012) 2173
29.0 PLB 855 (2024) 138764
36.1 PRD 97 (2018) 032005
20.3 JHEP 01 (2017) 099
4.6 JHEP 03 (2013) 128
139 JHEP 11 (2021) 169
20.2 PRD 95 (2017) 112005
4.9 JHEP 01, 086 (2013)
4.6 PRD 87, 112003 (2013)
36.1 JHEP 03 (2020) 054
20.3 PRD 93, 112002 (2016)
4.6 PRD 87, 112003 (2013)
140 JHEP 05 (2024) 131
20.3 JHEP 11 (2015) 172
140 arXiv:2312.04450
20.3 JHEP 11, 172 (2015)
140 arXiv:2403.09452
20.2 JHEP 11 (2017) 086
4.6 PRD 91 (2015) 072007
140 EPJC 83 (2023) 496
20.2 EPJC 77 (2017) 474
4.7 EPJC 77 (2017) 474
139 EPJC 81 (2021) 163
20.3 JHEP 04, 031 (2014)
20.2 PLB 781 (2018) 55
139 EPJC 83 (2023) 539
20.3 PRD 93, 112002 (2016)
140 PLB 848 (2024) 138400
20.3 PRL 115, 031802 (2015)
20.2 EPJC 77 (2017) 646
140 arXiv:2305.16994
139 PRL 129 (2022) 061803
79.8 PLB 798 (2019) 134913
140 PLB 846 (2023) 138222
20.3 JHEP 07 (2017) 107
140 arXiv:2403.02809
139 PLB 816 (2021) 136190
20.2 PRD 94 (2016) 032011
140 arXiv:2403.04869
139 JHEP 04 (2024) 026
20.3 PRD 96, 012007 (2017)
140 arXiv:2403.15296
20.3 PRD 93 (2016) 092004
139 Nature Phys. 19 (2023) 237

Despite these challenges, some observables at the LHC can be precisely predicted (and measured!). Key ingredients

- Factorization and asymptotic freedom. Short-distance QCD effects can be computed in perturbation theory
- Infrared safety: sufficiently inclusive observables are insensitive to long-distance hadronization effects.
- Modeling: parton shower Monte Carlo event generators do a great job at simulating realistic events, including hadronization.

Outline of the lectures

1. (Non-)perturbative QCD
    - Gauge invariance and Lagrangian
    - Feynman rules and perturbation theory
    - Asymptotic freedom
    - *R*-ratio and hadronization effects
2. Higher order corrections and IR safety
    - IR divergences and their cancellation
    - IR safety
    - Event shapes, jets, EECs

Outline (…)

3. Factorization, evolution, resummation
    - Soft and collinear factorization
    - Parton Distribution Functions (PDFs)
    - DGLAP evolution
    - The Drell-Yan process
4. Monte-Carlo techniques and parton showers
    - Monte Carlo techniques
    - Fixed-order results up to N^3LO
    - Parton showers
    - Modeling (Recoil, UE, hadronization, …)

It is better to uncover a little than to cover a lot.

Victor Weisskopf

Please stop me at any point during the lectures if you have questions!

QFT textbooks

- An Introduction to Quantum Field Theory, G. Sterman ‘93
- An Introduction to Quantum Field Theory, M. Peskin and D. Schroeder ’95
- Quantum Field Theory and Standard Model, M. Schwartz ’13
- …

Collider QCD

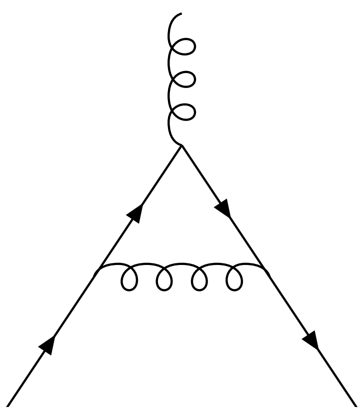
- QCD and Collider Physics, R. K. Ellis, W. J. Stirling, B. R. Webber '96
- The Black Book of Quantum Chromodynamics, J. Campbell, J. Huston, F. Krauss '17
- Quantum Chromodynamics, Huston, Rabbertz, Zanderighi, review by the Particle Data Group '25

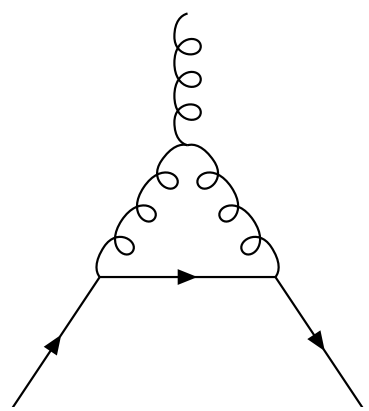
Special topics

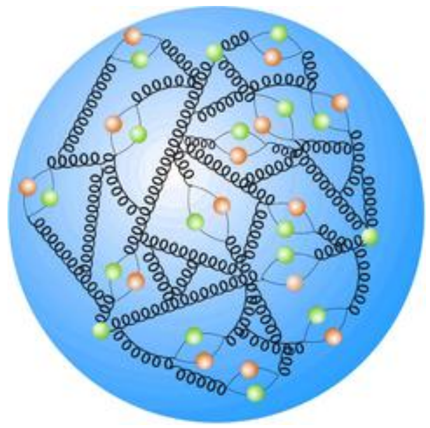
- Pythia 6.4 Physics and Manual, Sjostrand, Mrenna, Skands ’06 + manuals for later versions
- Towards Jetography, G. Salam ‘09
- Introduction to Soft-Collinear Effective Theory, T. Becher , A. Broggio , A. Ferroglia ’15
- Jet Substructure at the LHC: A Review of Recent Advances in Theory and Machine Learning, A. Larkoski, I. Moult, B. Nachman ‘17
- Energy Correlators: A Journey From Theory to Experiment, I. Moult and H.X. Zhu ‘25

Part I

(Non-)perturbative QCD







- Gauge invariance and Lagrangian
- Feynman rules and perturbation theory
- Asymptotic freedom
- R-ratio and hadronization effects

Gauge Theories

Theories with invariance under gauge transformations ("*local symmetry*")

$$\psi(x) \rightarrow \psi'(x) = V(x)\,\psi(x)$$

In QED the transformation $V(x) = \exp(i\alpha(x))$ is simply a space-time dependent phase factor.

Transformations form the abelian group $U(1)$:

$$V_2(x)\,V_1(x) = \exp\left(i\,[\alpha_1(x) + \alpha_2(x)]\,\right) \in U(1)$$

Non-abelian gauge theories

Yang and Mills '53 generalized concept to non-abelian gauge groups

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_N(x) \end{pmatrix} \to \psi'(x) = \mathbf{V}(x)\,\psi(x)$$

Vector of N fermion fields

$N \times N$ matrix

Matrices $\mathbf{V}(x)$ form a group such as $U(N)$ or $SU(N)$.

$SU(N)$ matrices fulfill

$$\mathbf{V}\,\mathbf{V}^\dagger = \mathbf{1} \quad \text{and} \quad \det(\mathbf{V}) = 1$$

and therefore have $N^2 - 1$ free parameters.
Parameterize group elements as exponentials

repeated indices summed!
sum over $a = 1, \ldots, N^2 - 1$

$$\mathbf{V} = e^{i\,\omega^a\,\mathbf{T}^a} = \mathbf{1} + i\,\omega^a\,\mathbf{T}^a + \ldots$$

group generators, $N \times N$ matrices, traceless, hermitian (exercise)

real parameters

Normalization: $\mathrm{tr}(\mathbf{T}^a\,\mathbf{T}^b) = \frac{1}{2}\delta^{ab}$

For products we use Baker-Campbell-Hausdorff

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\ldots}$$

higher commutators

$\neq 0$ for non-abelian group

To evaluate this, we need commutator of group generators

$$[\mathbf{T}^a, \mathbf{T}^b] \equiv i f^{abc}\, \mathbf{T}^c$$

"Lie Algebra"

structure constants,
fully determine group multiplication

Important examples

Pauli matrices

- SU(2): $\mathbf{T}^a = \frac{\sigma^a}{2}$; $f^{abc} = \varepsilon^{abc}$

- SU(3): $\mathbf{T}^a = \frac{\lambda^a}{2}$

Gell-Mann matrices

One rarely needs explicit form of the generators. Group theory factors in Feynman diagrams can be expressed through Casimir invariants, e.g.

$$\mathbf{T}^a\,\mathbf{T}^a = C_F\,\mathbf{1} \quad \text{with} \quad C_F = \frac{N^2-1}{2N} \quad \text{for SU(N)}$$

see Ritbergen, Schellekens, Vermaseren '98

Free fermion Lagrangian

Dirac Matrix

$$\mathscr{L} = \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi \equiv \sum_{i=1}^{N} \overline{\psi}_i \left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi_i$$

Mass term is gauge invariant

$$\overline{\psi}(x)\,\psi(x) \to \overline{\psi}(x)\,\mathbf{V}^{\dagger}(x)\,\mathbf{V}(x)\,\psi(x)$$

since $\mathbf{V}^{\dagger}\,\mathbf{V} = \mathbf{1}$, but kinetic Term is not

$$\overline{\psi}(x)\,\partial_{\mu}\psi(x) \to \overline{\psi}(x)\,\mathbf{V}^{\dagger}(x)\,\partial_{\mu}\,\mathbf{V}(x)\,\psi(x)$$

$$= \overline{\psi}(x)\,\partial_{\mu}\psi(x) + \overline{\psi}(x)\,\mathbf{V}^{\dagger}(x)\,[\partial_{\mu}\,\mathbf{V}(x)]\,\psi(x)$$

because derivative also acts on $\mathbf{V}(x)$!

Regular derivative cannot be invariant, since the fields at different x transform with different $\mathbf{V}(x)$. Need to introduce an extra gauge field that connects transformations at different points!

Define covariant derivative

$$\mathbf{D}_\mu = \partial_\mu \mathbf{1} - i\, g\, \mathbf{A}_\mu(x)$$

with $\mathbf{A}_\mu(x) = A_\mu^a(x)\, \mathbf{T}^a$. For $SU(N)$ we need to introduce $N^2 - 1$ vector fields $A_\mu^a(x)$ to achieve gauge invariance.

The covariant derivative of the fermion field transforms in the same way as the field itself

$$\mathbf{D}_\mu \psi(x) \rightarrow \mathbf{V}(x)\ \mathbf{D}_\mu \psi(x)$$

if the gauge field transforms as (exercise)

$$\mathbf{A}_\mu \rightarrow \mathbf{V}\mathbf{A}_\mu\mathbf{V}^\dagger - \frac{i}{g}\left(\partial_\mu \mathbf{V}\right)\mathbf{V}^\dagger$$

Can now easily construct gauge invariant terms using the covariant derivative!

Field strength tensor

Commutator of covariant derivatives

$$\mathbf{F}_{\mu\nu} = \frac{i}{g}\left[\mathbf{D}_\mu, \mathbf{D}_\nu\right] = \mathbf{T}^a F^a_{\mu\nu}$$

takes the form (exercise)

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu A^c_\nu$$

extra term for non-abelian group!

Kinetic term

Field strength tensor transforms as

$$\left[\mathbf{D}_\mu, \mathbf{D}_\nu\right] \rightarrow \mathbf{V}\left[\mathbf{D}_\mu, \mathbf{D}_\nu\right]\mathbf{V}^\dagger$$

but the trace

$$-\frac{1}{2}\mathbf{Tr}\ \mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu}$$

is invariant and provides a kinetic term for gauge bosons, along with boson self-interactions.

Gauge theory Lagrangian

$$\mathscr{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \overline{\psi} \Big(\gamma^\mu i\, \mathbf{D}_\mu - m\, \mathbf{1} \Big) \psi$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu A^c_\nu$$

$$i\, \mathbf{D}_\mu = i\, \partial_\mu \mathbf{1} + g\, \mathbf{T}^a A^a_\mu$$

Remarkably, both the electroweak and strong interactions are gauge theories!

Side-remarks: θ-term

Lagrangian on previous slide contains all terms up to operator dimension $d = 4$, with one exception

$$\mathscr{L}_\theta = \theta \frac{g_s^2}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} = \theta \frac{g_s^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu a}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor and θ is a free parameter. This term

- is a total derivative, not visible in PT
- violates P, T, CP

In QCD, term would induce e.g. an electric dipole moment neutron. Experimentally $\theta < 10^{-10}$.

Side-remarks: $d > 4$ terms

Can write down gauge invariant terms with higher dimension, for example

$$\Delta \mathscr{L} = \frac{1}{\Lambda^2} \mathrm{tr} \left(\mathbf{F}^{\mu}_{\ \nu} \mathbf{F}^{\nu}_{\ \rho} \mathbf{F}^{\rho}_{\ \mu} \right) \qquad (d = 6)$$

Higher-dimensional operators are suppressed by powers of scale Λ

- New heavy particles with masses $M \sim \Lambda$ induce such operators at low energies through virtual effects
- Precision measurements probe such operators!
- Systematic framework: SMEFT (ask Jason Aebischer!)

Side-remarks: Gauge fixing

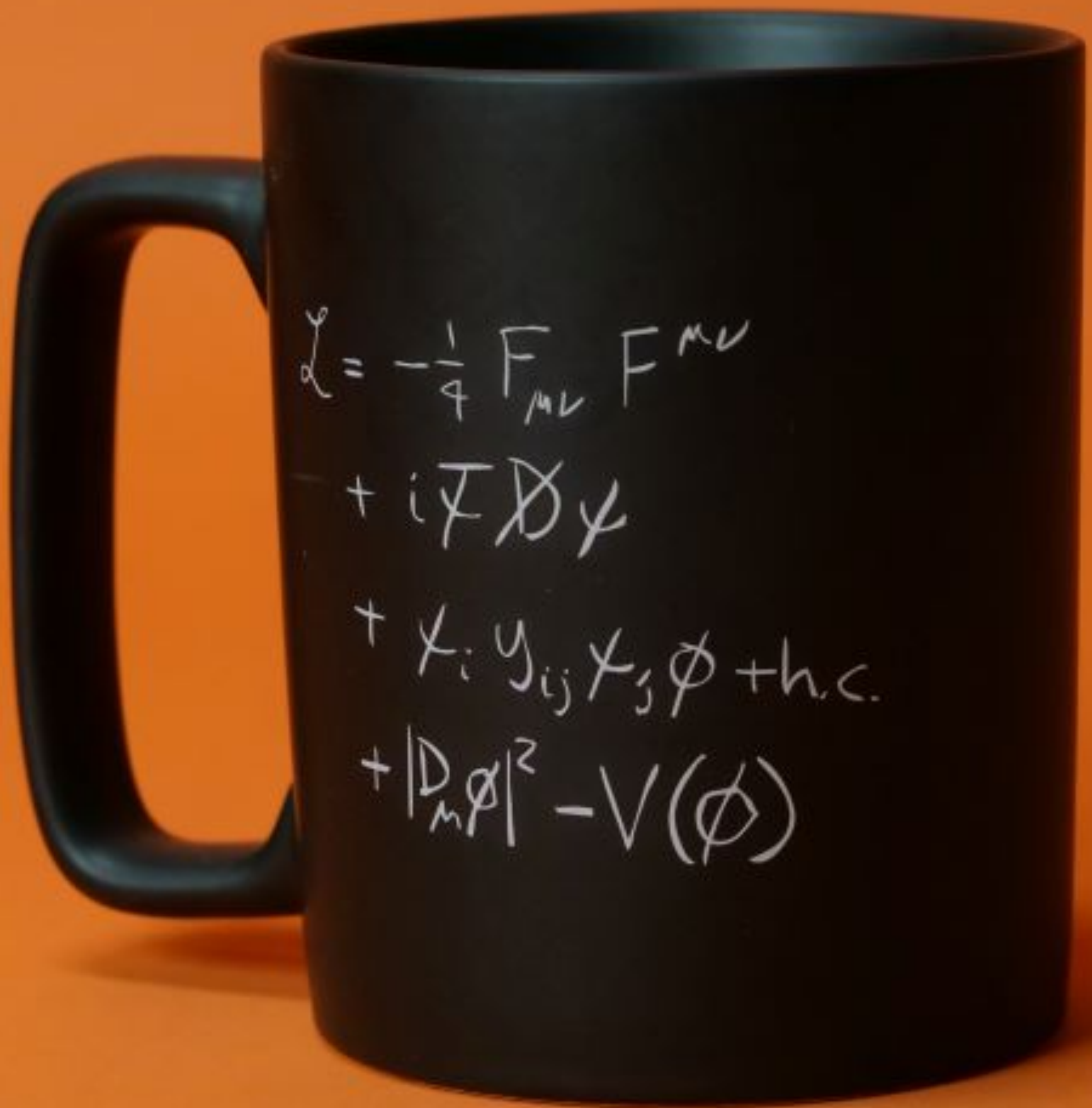
Due to the gauge symmetry many field configurations are equivalent. Drop out when computing expectation values, but cause problems in path integral

$$Z = \int \mathscr{D}\mathbf{A}_\mu \exp\left(iS[\mathbf{A}_\mu]\right)$$

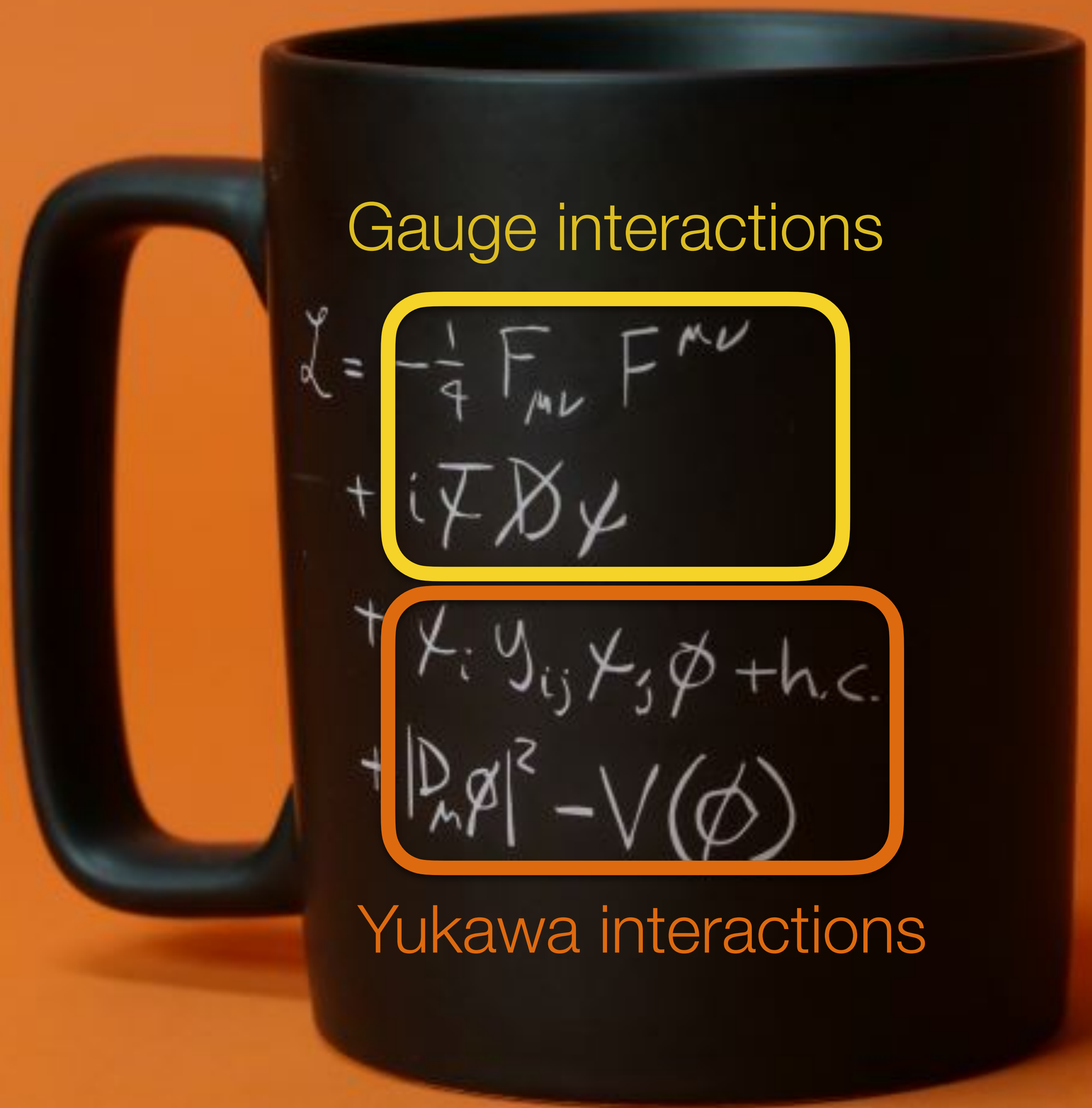
Solution by Faddeev-Popov '67 is to factor out integration over gauge-related configurations, leaving behind a gauge-fixed action.

Gauge fixing introduces extra terms into the action and auxiliary ``ghost" fields. Depending on gauge fixing, ghost fields enter higher-order computations.

Ghost fields will not be needed for this lecture, see QFT textbooks for more information.



Note: no mass terms! Masses are generated through vacuum expectation value of Higgs field ϕ.



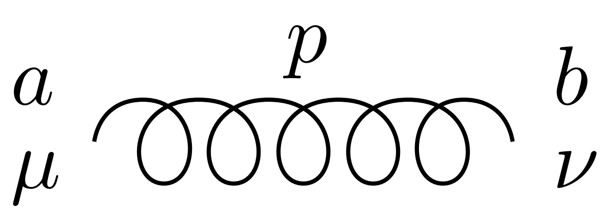
Note: no mass terms! Masses are generated through vacuum expectation value of Higgs field ϕ.

	Gauge group	Charged fermions	Gauge bosons	Coupling	Low-E
Strong interaction (QCD)	SU(3)	quarks u, d, c, s, t, b in $N_c = 3$ colors	$N_c^2 - 1 = 8$ gluons	$\alpha_s = \frac{g_s^2}{4\pi}$	Confinement
Electroweak	SU(2) x U(1)	all fermions different charges for ψ_L and ψ_R	$W^\pm, Z, \gamma$	$\alpha = \frac{e^2}{4\pi}$ G_F	Higgs mechanism, screening

Feynman rules for QCD

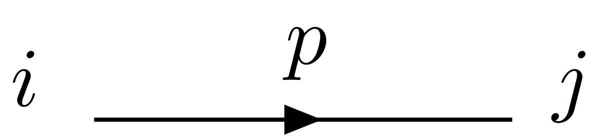
By expanding the action in the fields and Fourier transforming, one obtains Feynman rules in momentum space.

Bilinear terms in action give propagators



Gluon: $-i\,\delta_{ab}\,g_{\mu\nu}/p^2$ $\quad a, b = 1, 2, \ldots, N_c^2 - 1$

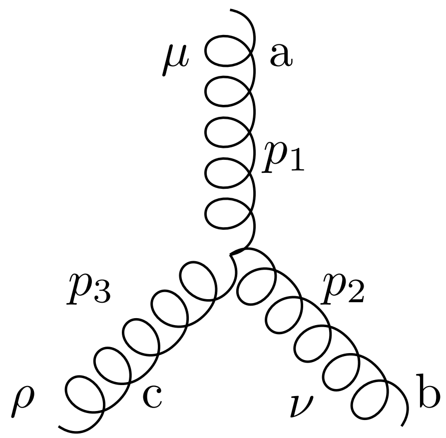
Feynman gauge. Different form for other gauge fixing



Fermion: $i\,\delta_{ij}(\gamma^\mu p_\mu + m)/(p^2 - m^2)$

$i, j = 1, \ldots N_c$

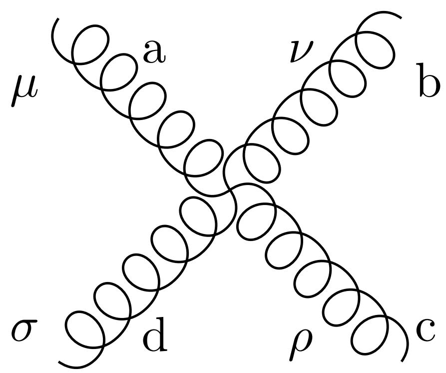
Interaction vertices



Lagrangian: $-g\, f^{abc}(\partial_\mu A_\nu^a)A_\mu^b A_\nu^c$

$$g_s\, f^{abc} \left(g_{\mu\nu} \left(p_1 - p_2\right)_\rho + g_{\nu\rho} \left(p_2 - p_3\right)_\mu + g_{\rho\mu} \left(p_3 - p_1\right)_\nu \right)$$

$3! = 6$ terms from symmetrization in gluon fields



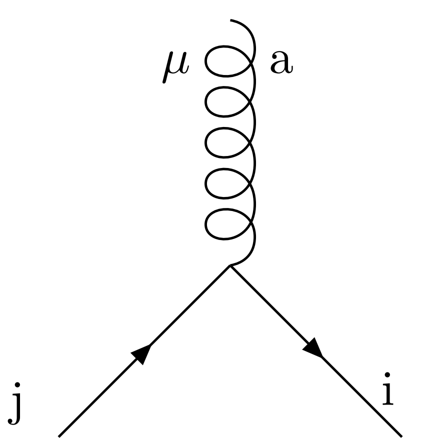
$$-i\, g_s^2 f^{eab} f^{ecd} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right)$$

$$-i\, g_s^2 f^{eac} f^{ebd} \left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho}\right)$$

$$-i\, g_s^2 f^{ead} f^{ebc} \left(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}\right)$$

$$\frac{1}{4} g_s^2 f^{abc} f^{ade} A_\mu^b A_\nu^c A_\mu^d A_\nu^e$$

$4! = 24$ terms, 4 identical

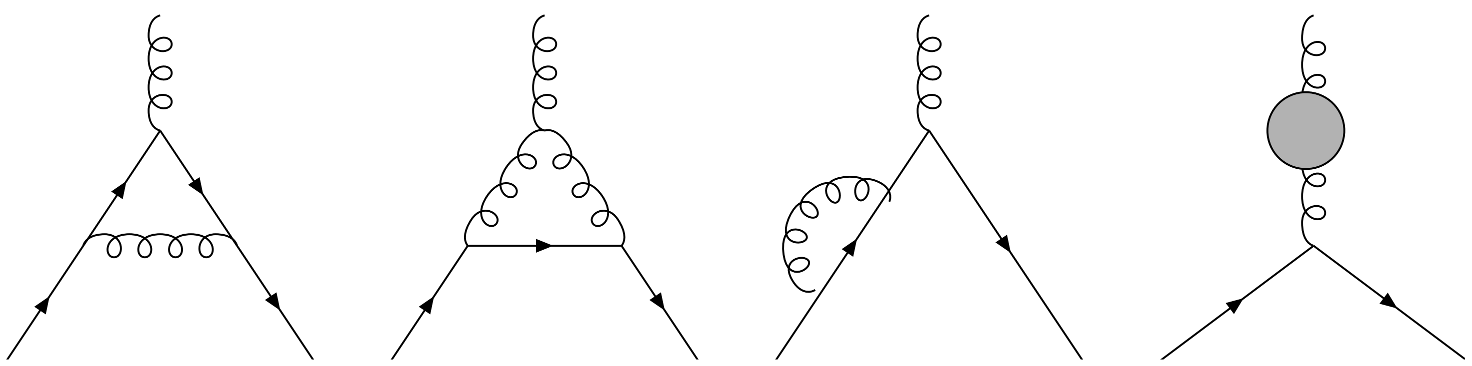


$$i\, g_s\, \gamma^\mu \left(T^a\right)_{ij}$$

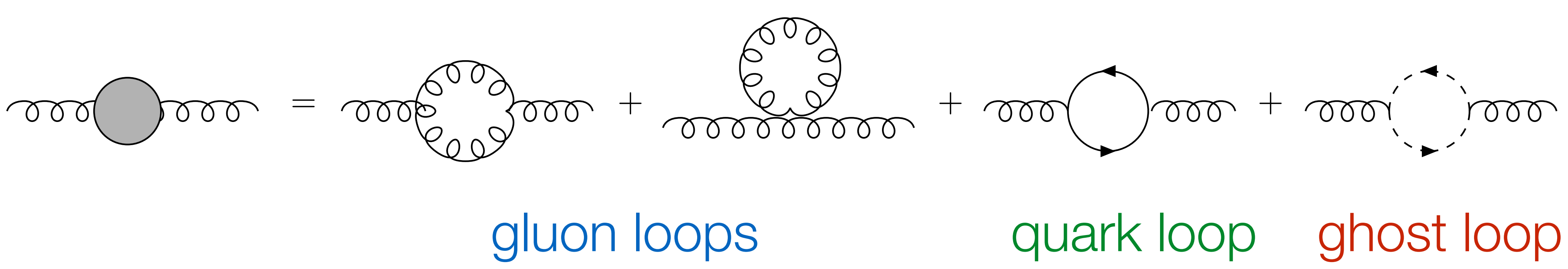
$$g_s \overline{\Psi}_i \gamma^\mu A_\mu^a (T^a)_{ij} \Psi_j$$

Loop corrections

Compute higher-order corrections to the quark-gluon coupling



with



Renormalization

Loop corrections suffer from UV divergences

- Regularize integrals (UV cutoff, dimensional regularization, …) to make divergences explicit
- Renormalization: Subtract divergent pieces and absorb them into parameters of theory, e.g.

subtraction scale *aka* renormalization scale

$$\alpha_s^0 \quad \to \quad \alpha_s(\mu)$$

bare coupling, absorbs divergences

renormalized coupling, finite

Running coupling

Behavior of the coupling when the scale μ is changed is governed by renormalization group equation

$$\mu \frac{\partial \alpha_s(\mu)}{\partial \mu} = \frac{\partial \alpha_s(\mu)}{\partial \ln \mu} = \beta\big(\alpha_s(\mu)\big)$$

driven by the β-function

$$\beta(\alpha_s) = -2\alpha_s \left[\beta_0 \alpha_s + \beta_1 \alpha_s^2 + \mathcal{O}(\alpha_s^3)\right]$$

from one-loop diagrams   from two-loop diagrams

Solution at one loop

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1+\alpha_s(\mu_0)\,\beta_0 \ln(\mu^2/\mu_0^2)}$$

value at reference scale μ_0

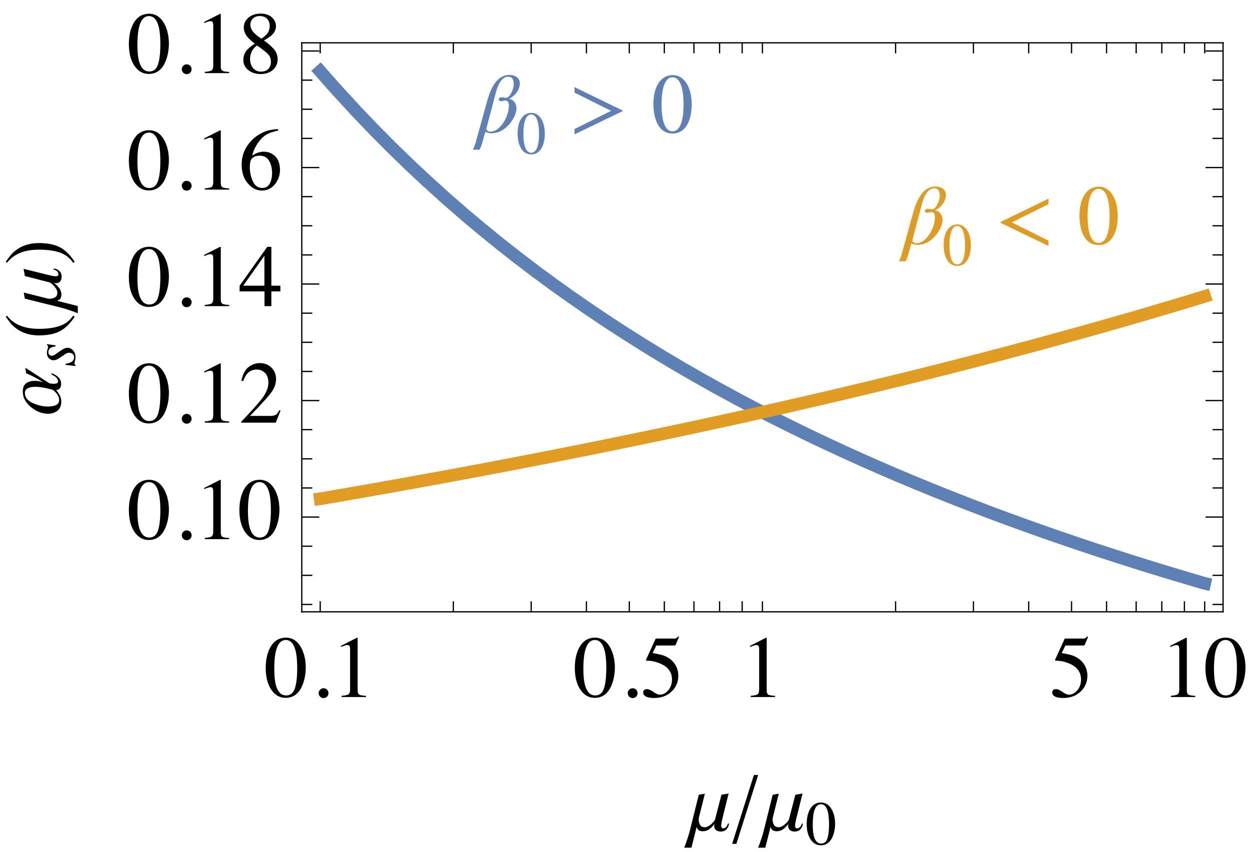
In QCD one obtains

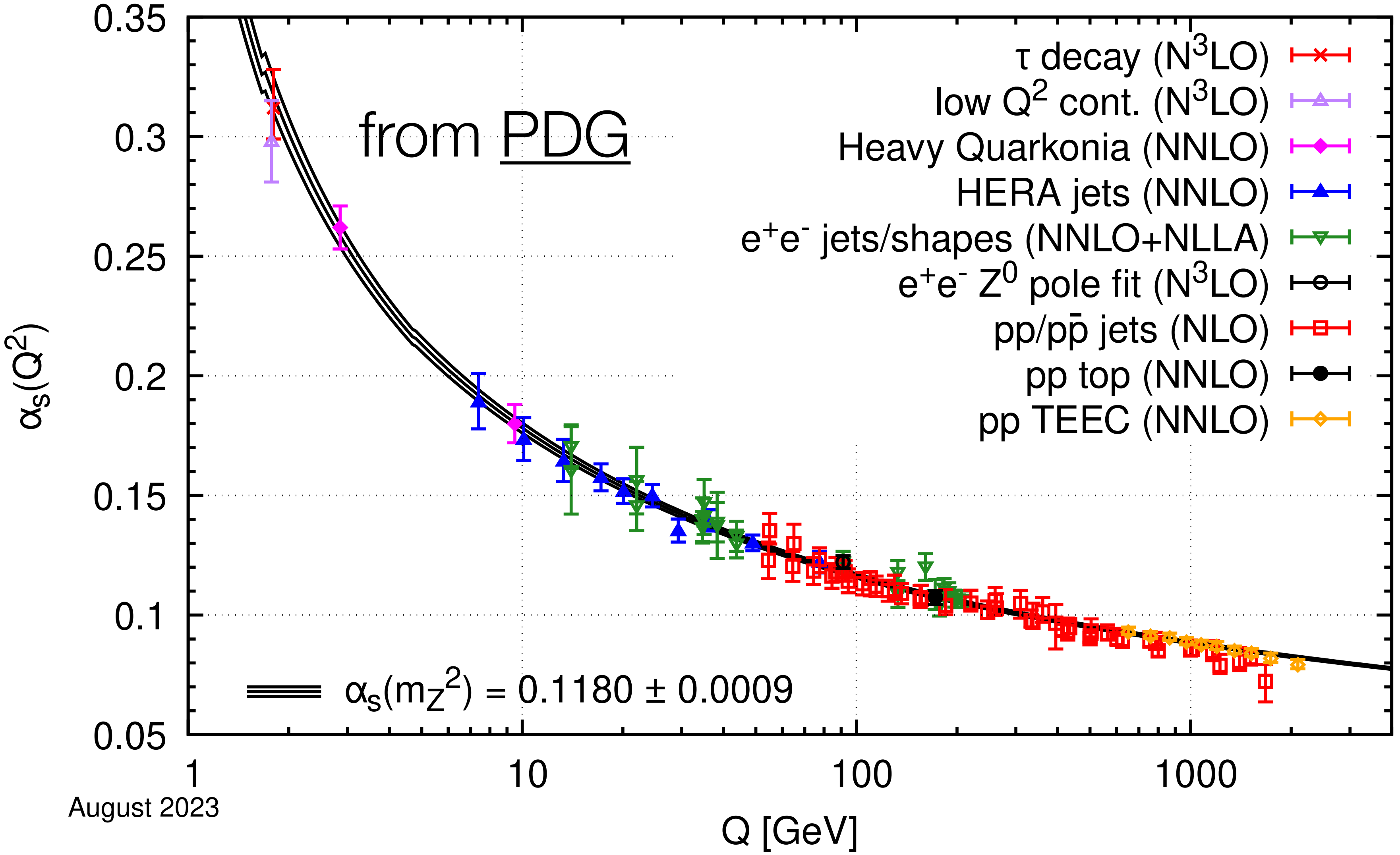
$$\beta_0 = \frac{1}{12\pi}\left(11\,N_c - 2n_f\right) > 0$$

weak coupling at very high energies:

asymptotic freedom!

Nobel Prize ’04
for Gross, Politzer, Wilczek



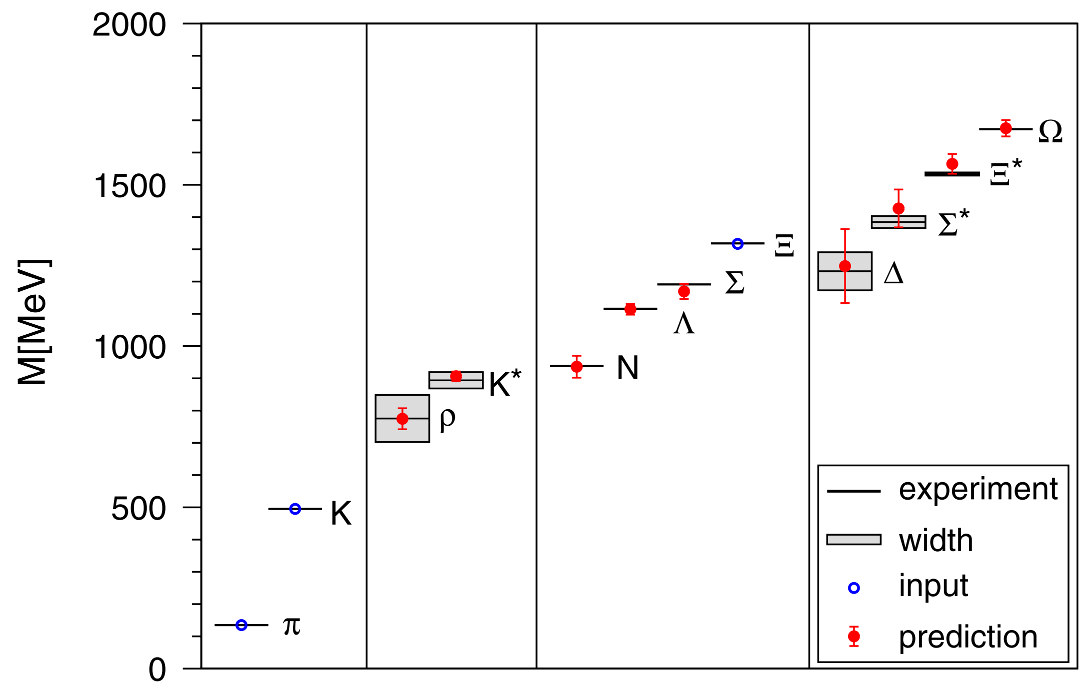


- Running of coupling confirmed by experimental measurements at different energies with $\mu = Q$

- Coupling $\alpha_s(\mu) \to \infty$ at low $\mu = \Lambda_{\mathrm{QCD}} \sim 200\,\mathrm{MeV}$

- Note: β-function has been computed to 5 loops! Implemented in code RunDec.

Low energy: non-perturbative QCD



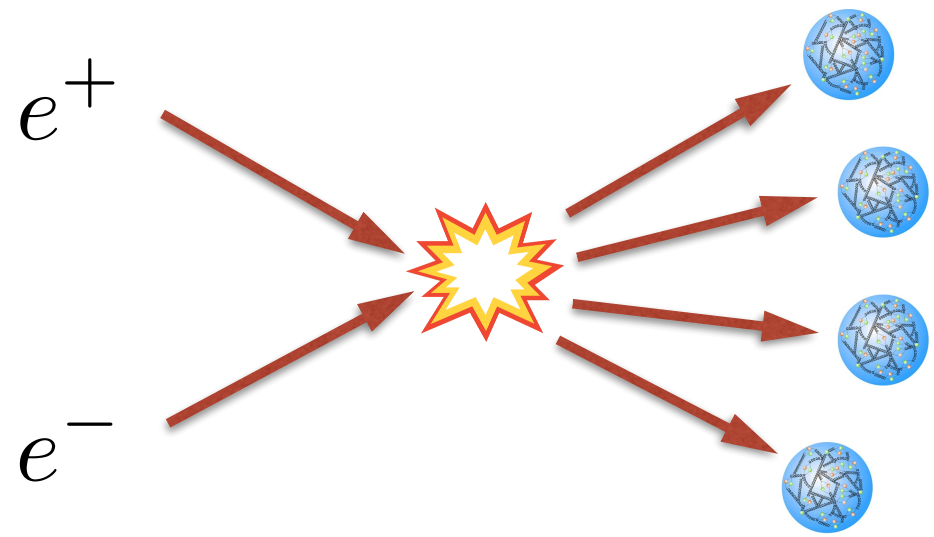
Dürr et al. '08
BMW Collaboration

Numerical solution of QCD path integral with lattice QCD successfully determines simple ("Euclidean") low-energy quantities

- hadron masses, hadron form factors, …

Side remark: Proton and neutron masses almost entirely due to non-perturbative QCD dynamics, quark mass contribution (due to Higgs VEV) very small.

Perturbative QCD?



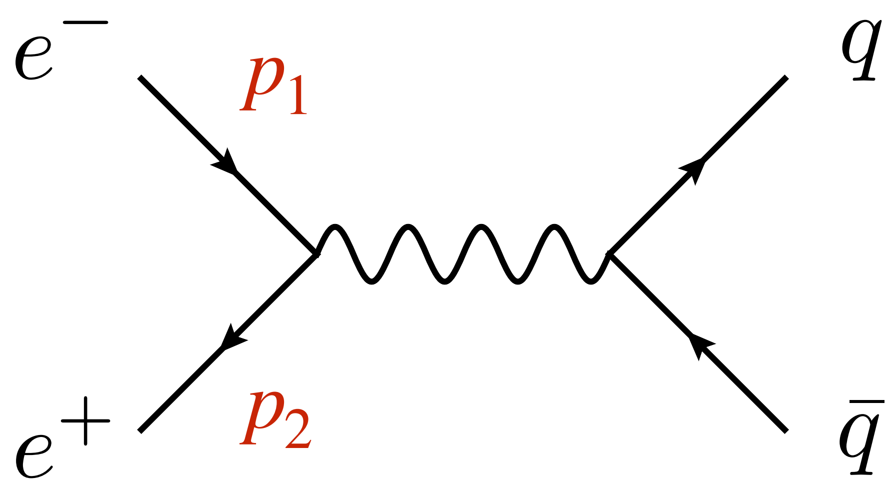
Let us now compute the inclusive cross section

$$\sigma(e^+e^- \to \text{hadrons})$$

in perturbation theory, by boldly replacing the final state with quarks and gluons.

R-ratio

Lowest order diagram



has the same form as $e^+e^- \to \mu^+\mu^-$. Define the ratio [$s = (p_1 + p_2)^2$]

$$R(s) = \frac{\sigma(e^+e^- \to q\,\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

Theoretical prediction

If we neglect quark and muon masses, numerator and denominator are identical up to charge factors.

$$R(s) = N_c \sum_f Q_f^2$$

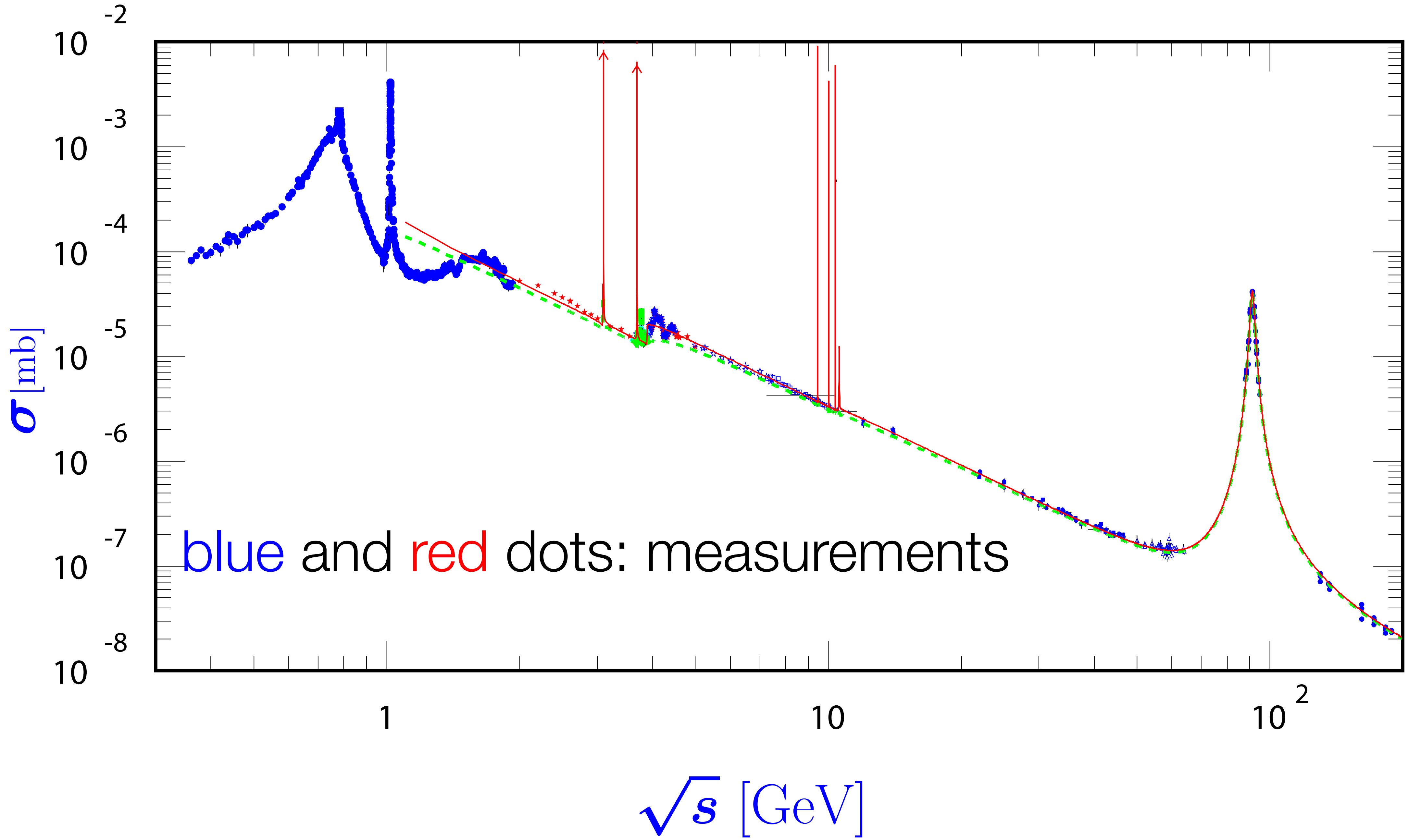
$N_c = 3$ quarks per flavor

$Q_f = +2/3$ for u, c, t

$Q_f = -1/3$ for d, s, b

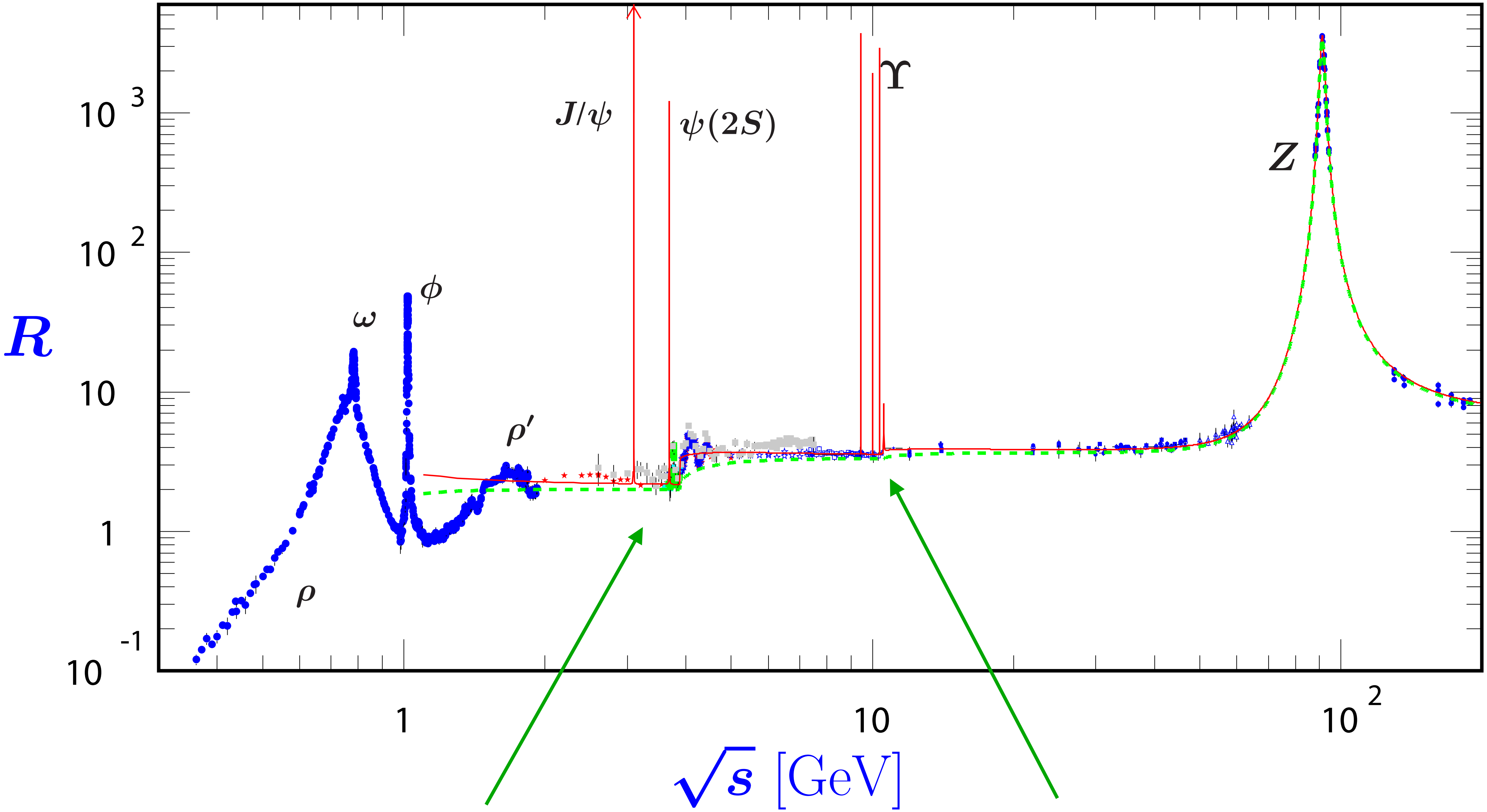
sum over all quark flavors f
accessible at center-of-mass energy s

$e^+e^- \to$ hadrons: cross section



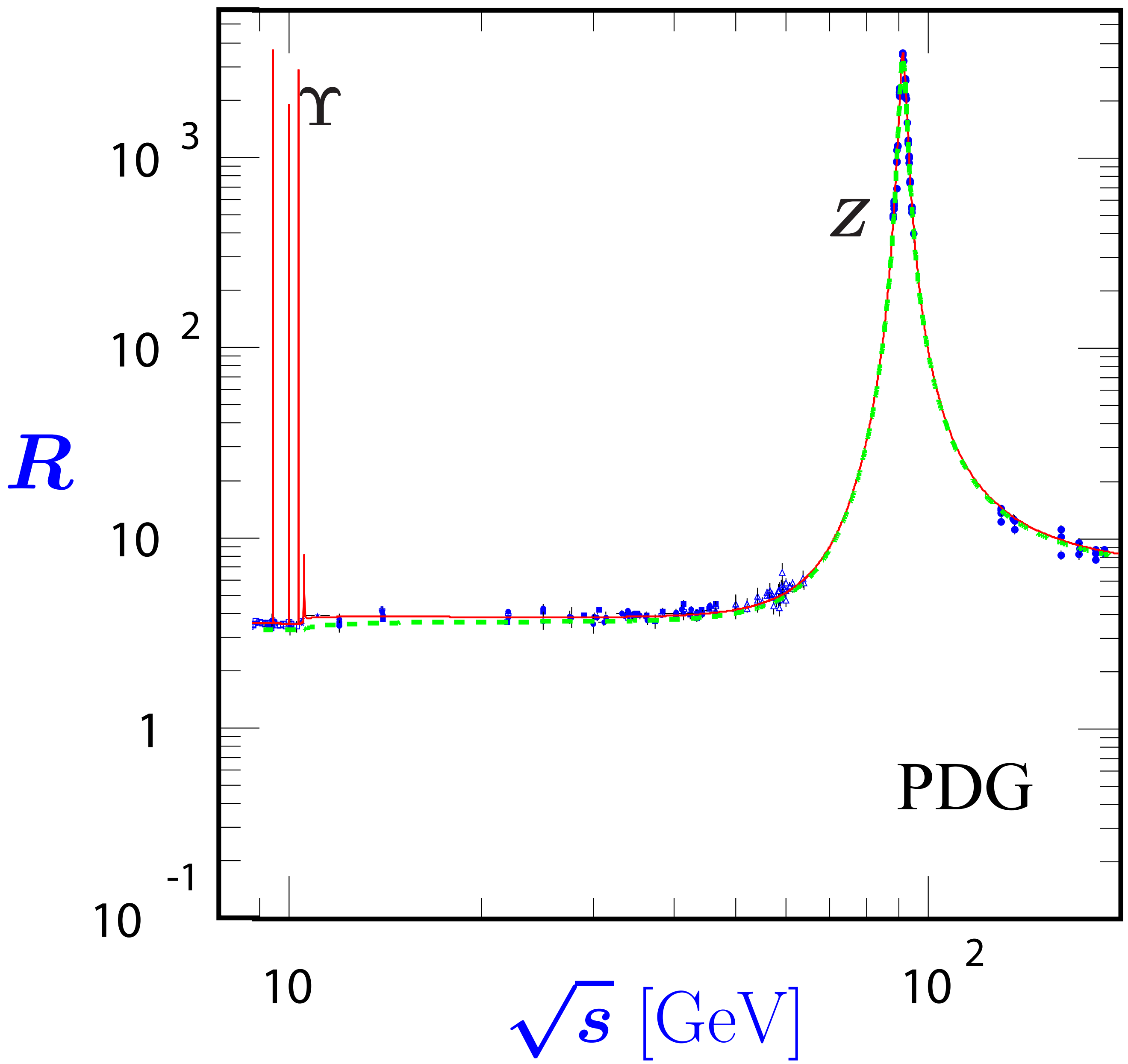
compiled by the Particle Data Group

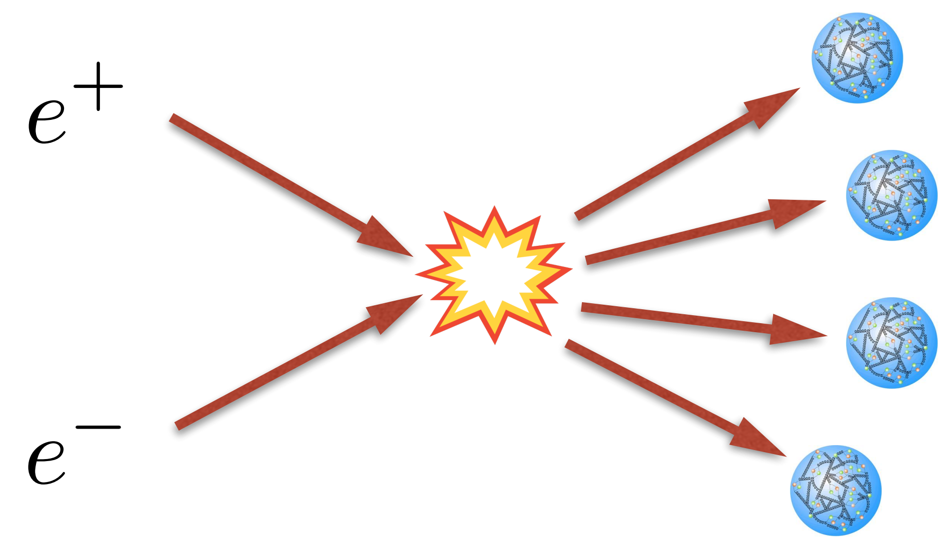
R-ratio



$c\bar{c}$ resonances
$m_c \approx 1.3\,\text{GeV}$

$b\bar{b}$ resonances
$m_b \approx 4.2\,\text{GeV}$





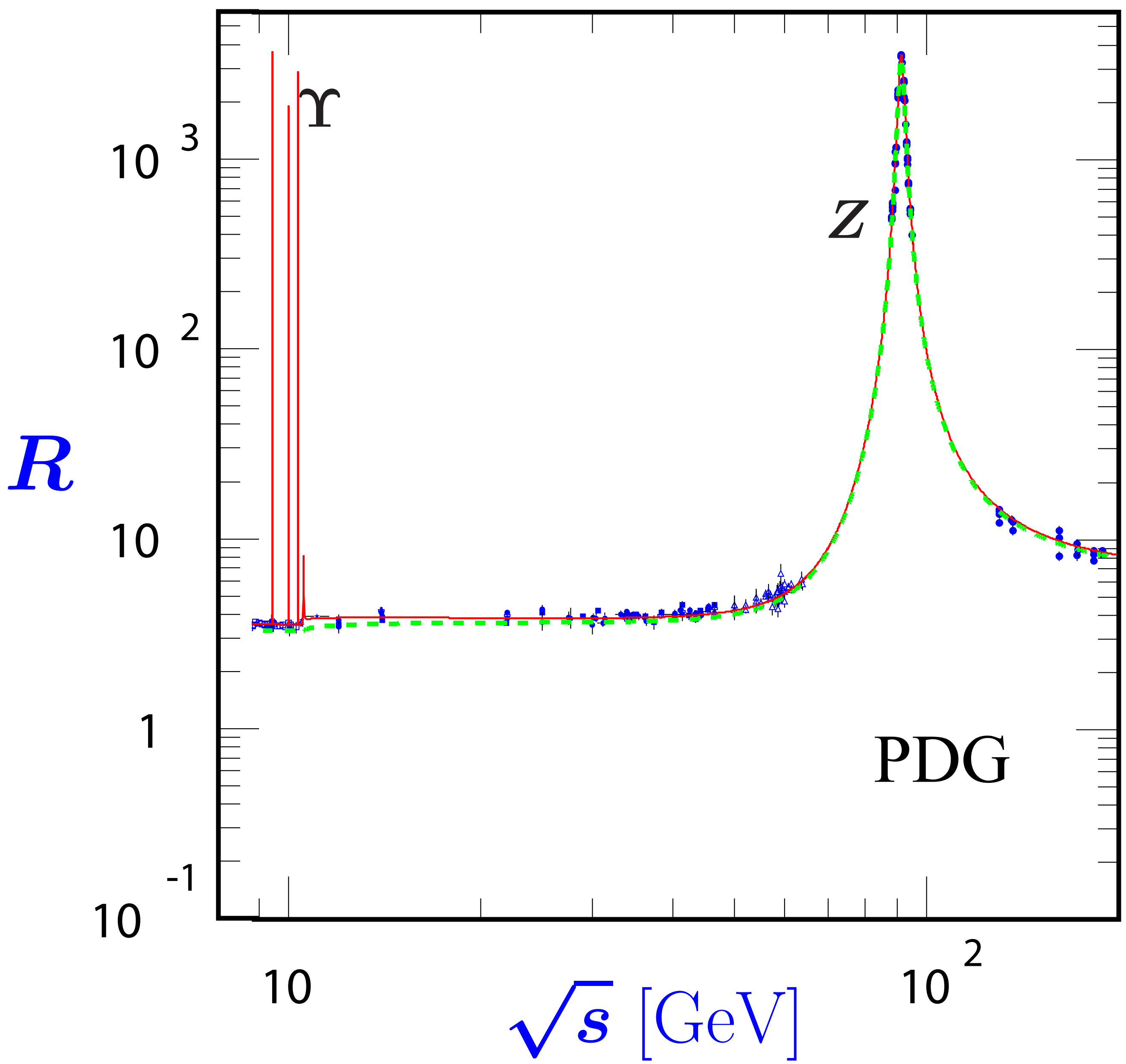
measurement

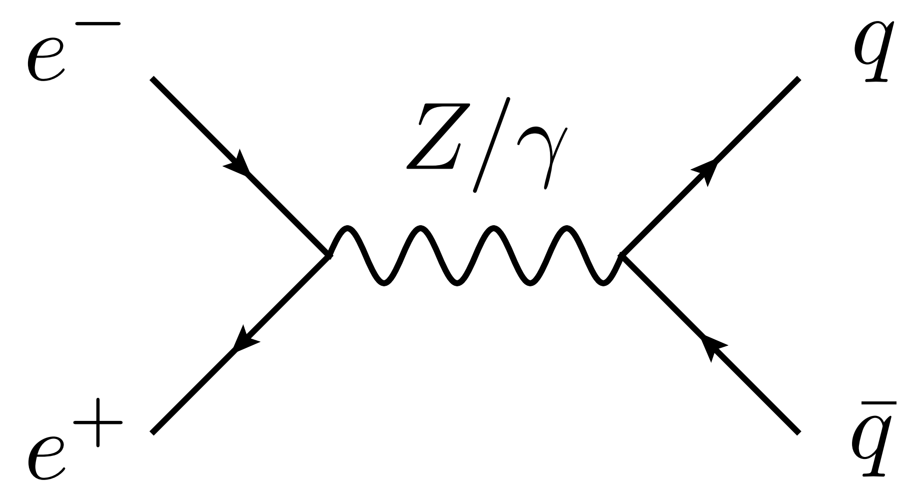
$$R = \frac{\sigma(e^+e^- \to Z/\gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

theoretical expression
w/o Z-boson exchange

Blue: experimental measurements

Green and red lines theoretical predictions





sum over colors and flavors of quarks

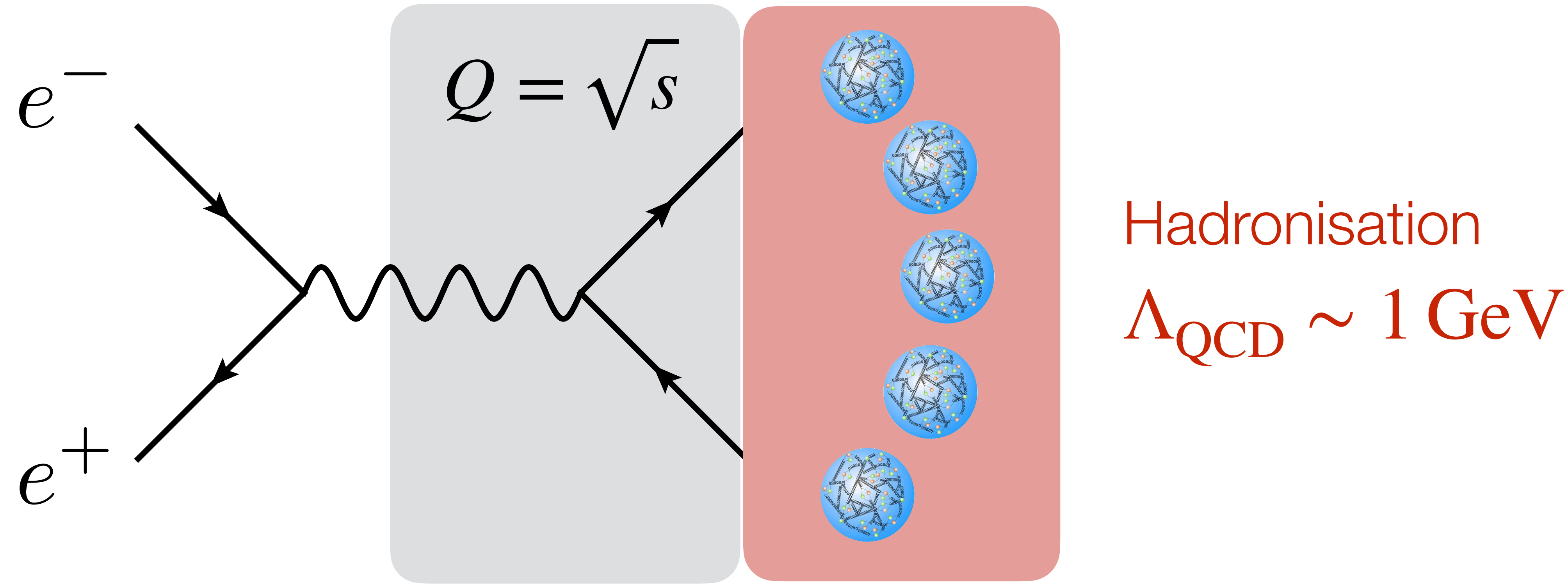
$$R_{\text{pert}} = \frac{\sigma(e^+e^- \to Z/\gamma^* \to q\bar{q})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

Dashed green: LO perturbation theory

Solid red: N^3LO perturbation theory

Remarkable agreement with data: asymptotic freedom

Intuitive explanation:



Large scale separation $Q \gg \Lambda_{\rm QCD}$.

- Two step process: 1.) $q\bar{q}$ production 2.) rearrangment into hadrons
- For $\sigma_{\rm total}$, small sensitivity to step 2.)

Formal explanation: the **Operator Product Expansion (OPE)*** factorizes low and high energy contributions

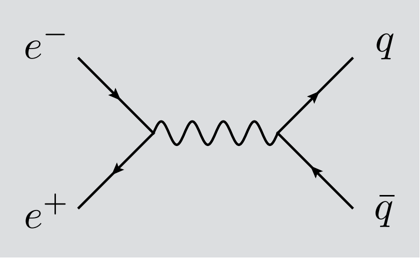
$$R(s) = C_1(s)\, \langle 0|\, 1\, |0\rangle + C_{q\bar{q}}(s)\, \langle 0|\, m_q \bar{q} q\, |0\rangle + C_{GG}(s)\, \langle 0|\, G^2\, |0\rangle + \ldots$$

$\langle 0|\, 1\, |0\rangle = 1$

$\langle 0|\, m_q \bar{q} q\, |0\rangle \sim m_q\, \Lambda^3_{\text{QCD}}$

$\langle 0|\, G^2\, |0\rangle \sim \Lambda^4_{\text{QCD}}$

$$\sim 1/s^2$$



Wilson coefficients:
high-energy physics
independent of states

Matrix elements:
non-perturbative,
hadronisation effects

* see e.g. Peskin & Schröder p.615

Part II

Higher-order corrections and IR safety

- IR divergences and their cancellation
- IR safety
- Event shapes, jets, EECs

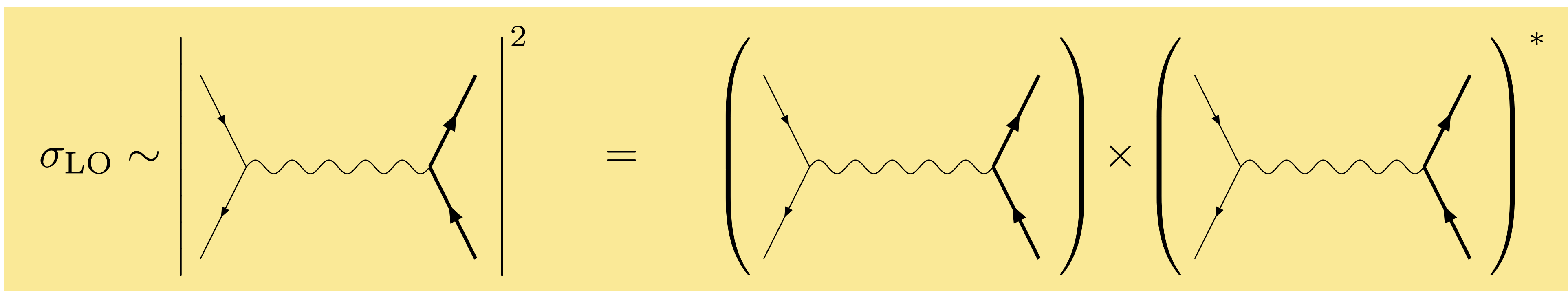
The successful prediction of the R-ratio in perturbation theory leads to the following questions

1. Can one improve the prediction by going to higher orders in perturbation theory?
2. Are there other, less inclusive cross sections, which are insensitive to hadronisation effects?

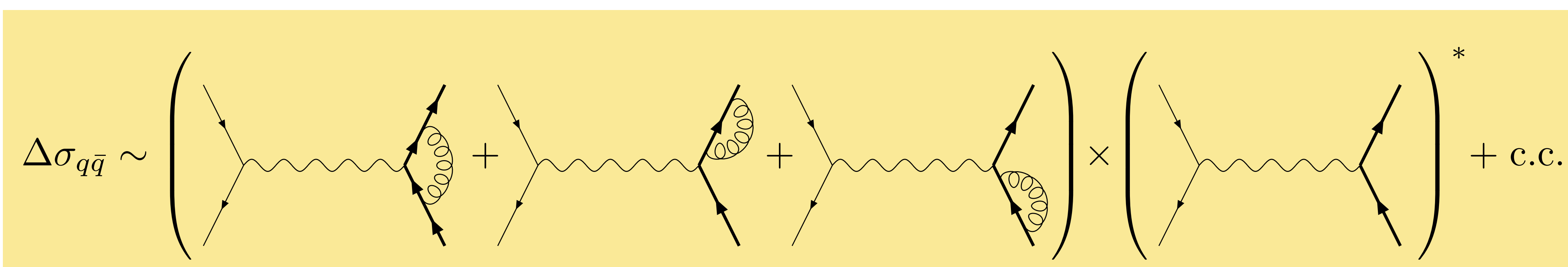
Interestingly, answer to 1.) informs 2.). Will first study the structure of perturbative corrections, then introduce classes of observables which are insensitive to hadronisation.

Perturbative corrections

In the previous lecture we computed the leading order (LO) *R*-ratio



The loop corrections are



Loop integrals suffer from divergences. Regularize them by computing in $d = 4 - 2\varepsilon$ (dimensional regularization).

Result ($Q^2 = s$)

$$\Delta\sigma_{q\bar{q}} = \sigma_{\mathrm{LO}} \frac{\alpha_s}{3\pi} \left(\frac{\bar{\mu}^2}{Q^2} \right)^{\varepsilon} \left(-\frac{4}{\varepsilon^2} - \frac{6}{\varepsilon} - 16 + \frac{7}{3}\pi^2 + \mathcal{O}(\varepsilon) \right)$$

[$\bar{\mu}^2 = 4\pi\, e^{\gamma_E} \mu^2$]

diverges for $\varepsilon \to 0$!

These are not ultraviolet divergences!

Repeat the computation with massive quarks and gluons.

Result for small masses is finite (source: ChatGPT)

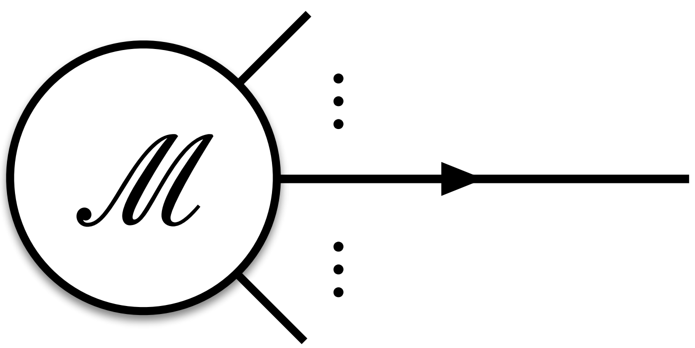
$$\Delta\sigma_{q\bar{q}} = \sigma_{\mathrm{LO}}\frac{\alpha_s}{3\pi}\left[-2\ln^2\left(\frac{Q^2}{m_q^2}\right) + 6\ln\left(\frac{Q^2}{m_q^2}\right) - 2\ln\left(\frac{Q^2}{m_g^2}\right) + \text{constants}\right]$$

but depends on small quark masses and unphysical gluon mass.

Here masses act as infrared regulator: divergences come back as we switch off the masses!

Divergences arose because we computed the unphysical exclusive cross section $\sigma_{q\bar{q}}$.

In theories with massless particles (QED, QCD, …) fully exclusive cross sections do not make sense! For massless particles cannot distinguish



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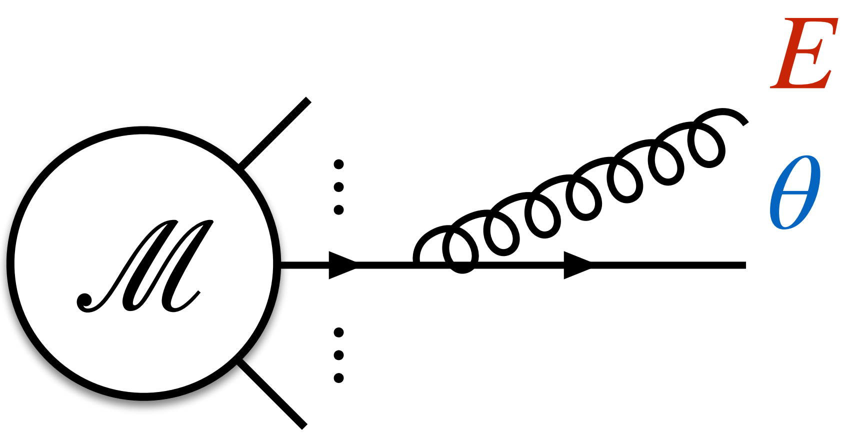


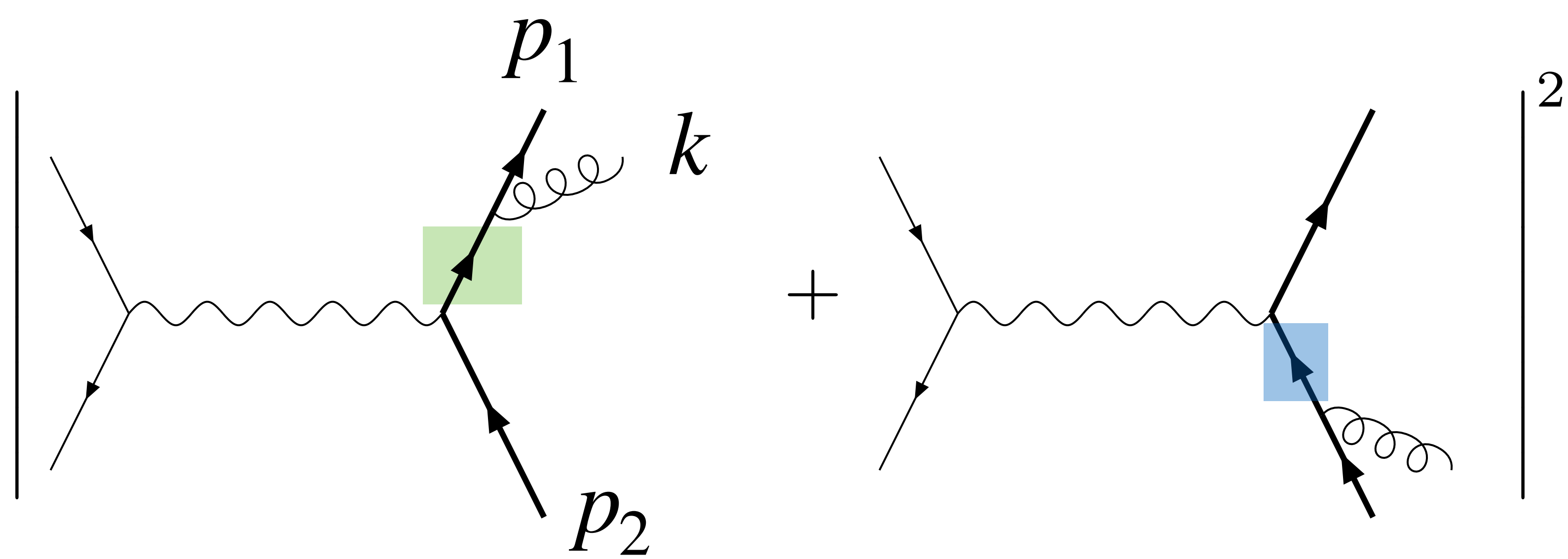
if emission is soft ($E \to 0$) or collinear ($\theta \to 0$) !

Bloch and Nordsieck ‘37

Kinoshita ’62 Lee, Nauenberg ‘64

Need to include real emission corrections!

$\Delta\sigma_{q\bar{q}g} =$



$$\frac{1}{4}\sum_{\text{spins}} |\mathscr{M}_{q\bar{q}g}|^2 = \sigma_{\text{LO}}\frac{16\pi}{Q^2}\, C_F\, g_s^2 \frac{(p_1\cdot k)^2 + (p_2\cdot k)^2 + Q^2 p_1\cdot p_2}{p_1\cdot k\; p_2\cdot k}$$

diverges for $k \to 0$ and for $k \parallel p_1$ or $k \parallel p_2$.

Phase space integral does not exist, regularize in $d = 4 - 2\varepsilon$.

Aside: phase-space in *d*-dimensions

Massless toy example, $k \equiv E_k = |\vec{k}|$

$$I = \int \frac{d^{d-1}k}{2E_k} \frac{1}{E_k^2} \theta(Q - E_k) = \int_0^Q dk\, k^{d-2} \int d\Omega_{d-1} \frac{1}{2E_k^3}$$

set $d = 4 - 2\varepsilon$

spherical coordinates

$$I = \frac{1}{2} \int_0^Q dk\, k^{-1-2\varepsilon}\, \Omega_{d-1} = -\frac{Q^{-2\varepsilon}}{4\varepsilon} \Omega_{3-2\varepsilon}$$

surface of *d*-1 dimensional unit sphere, see QFT books!

IR divergence

Rewrite kinematics in terms of variables y_i

$$2p_1 \cdot p_2 = y_3\, Q^2$$

$$2p_1 \cdot p_k = y_2\, Q^2$$

$$2p_2 \cdot p_k = y_1\, Q^2$$

In CMS system

$$q^\mu = p_1^\mu + p_2^\mu + k^\mu = (\, Q, 0, 0, 0\,)$$

and

$$y_i = 1 - \frac{2E_i}{Q} > 0 \quad \text{with} \qquad y_1 + y_2 + y_3 = 1$$

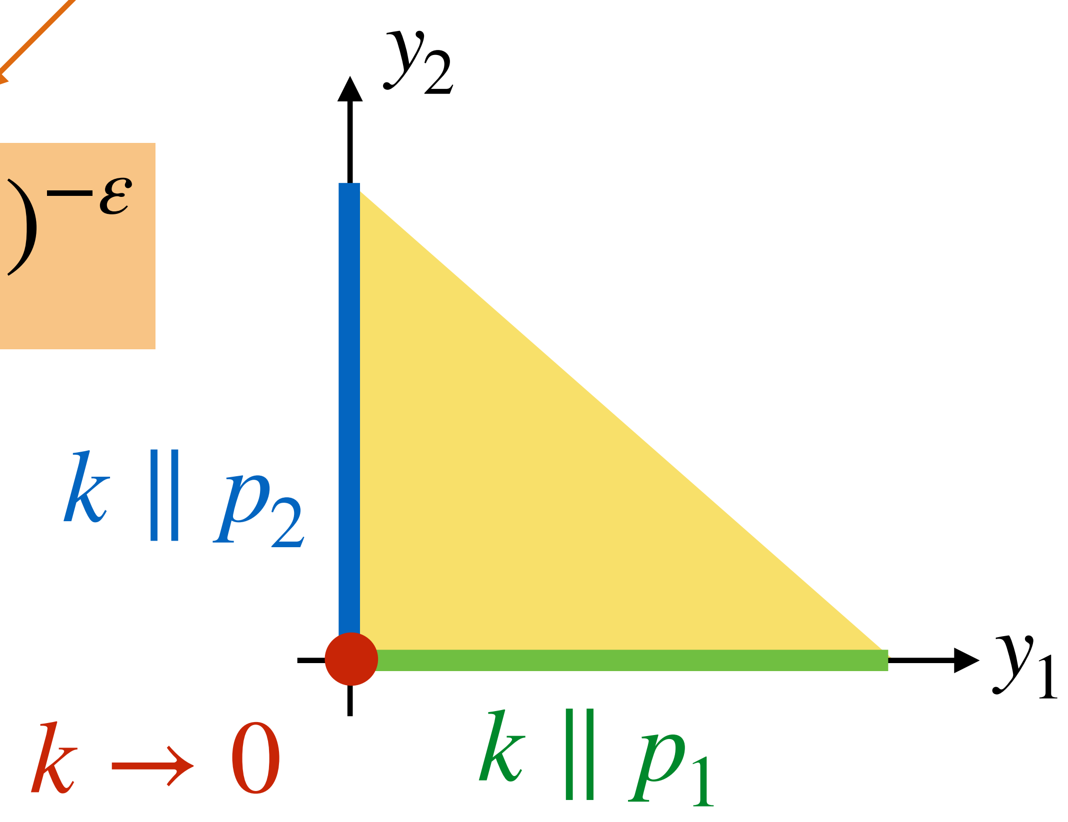
In terms of new variables (with $y_3 = 1 - y_1 - y_2$)

$$\frac{1}{4}\sum_{\text{spins}} |\mathscr{M}_{q\bar{q}g}|^2 = \sigma_{\text{LO}} \frac{16\pi}{Q^2} C_F\, g_s^2\, \frac{y_1^2 + y_2^2 + 2\,y_3}{y_1\, y_2} + O(\varepsilon)$$

Phase-space integral

$$PS_3 \propto \int_0^1 dy_1 \int_0^{y_1} dy_2\, (y_1 y_2 y_3)^{-\varepsilon}$$

regularized if $\varepsilon < 0$



With the regularization in place, we can compute the total cross section

$$\sigma_{\text{tot}} = \sigma(e^+e^- \to X) = \sigma_{q\bar{q}} + \sigma_{q\bar{q}g} + \mathcal{O}(\alpha_s^2)$$

and obtain

virtual corrections

$$\sigma_{\text{tot}} = \sigma_{LO}\left(1 + \frac{\alpha_s}{3\pi}\left[-\frac{4}{\varepsilon^2} - \frac{6}{\varepsilon} - 16 + \frac{7\pi^2}{3}\right] + \frac{\alpha_s}{3\pi}\left[\frac{4}{\varepsilon^2} + \frac{6}{\varepsilon} + 19 - \frac{7\pi^2}{3}\right]\right)$$

real emission

With the regularization in place, we can compute the total cross section

$$\sigma_{\text{tot}} = \sigma(e^+e^- \to X) = \sigma_{q\bar{q}} + \sigma_{q\bar{q}g} + \mathcal{O}(\alpha_s^2)$$

and obtain

few %

$$\sigma_{\text{tot}} = \sigma_{\text{LO}}\left(1 + \frac{\alpha_s(\mu)}{\pi}\right)$$

Finite! Small correction, insensitive to low-energy scales such as quark masses.

And excellent agreement with data far away from resonance regions!

Scale (in-)dependence

Perturbative result for the R-ratio

$$\sigma_{\text{tot}} = \sigma_{\text{LO}} \left(1 + \frac{\alpha_s(\mu)}{\pi} \right)$$

seems to depends on renormalization scale μ!

But σ_{tot} is a physical cross section, cannot depend on unphysical scale μ!

- change in μ is higher-order effect, compensated by perturbative corrections!

NNLO result for the *R*-ratio

$$\sigma_{\text{tot}} = \sigma_{\text{LO}} \left(1 + \frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi} \right)^2 \left(\pi \beta_0 \ln \frac{\mu^2}{Q^2} - 11\zeta_3 + \frac{365}{24} \right) + \left(\frac{2\zeta_3}{3} - \frac{11}{12} \right) n_f \right)$$

- β_0-terms compensate scale dependence of NLO coefficient
- Residual scale dependence is N^{3}LO effect
- Must choose $\mu \sim Q$ to avoid large logarithm in perturbative corrections

Since scale dependence is a higher-order effect, variation $Q/2 < \mu < 2\,Q$ is used to estimate perturbative uncertainty

IR finiteness

What other observables $\mathcal{O}$, defined in terms of the particle momenta $\{p_1, p_2, \ldots, p_n\}$ can be computed in perturbation theory? Observable must be

A. insensitive to soft radiation

$$\lim_{k \to 0} \mathcal{O}_{n+1}(p_1, p_2, \ldots, p_n, k) = \mathcal{O}_n(p_1, p_2, \ldots, p_n)$$

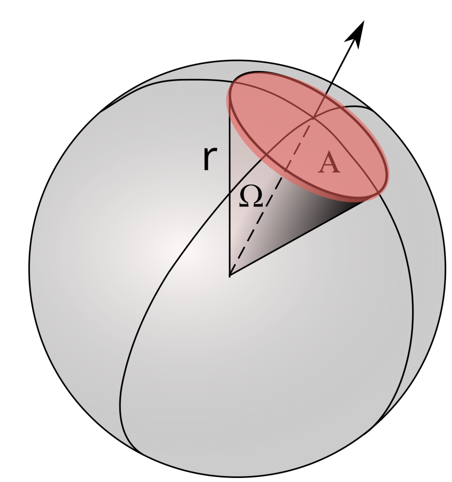
B. collinear safe for $p_1 \parallel p_2$

$$\mathcal{O}_{n+1}(p_1, p_2, \ldots, p_{n+1}) = \mathcal{O}_n(p_1 + p_2, \ldots, p_n)$$

If A.) and B.) hold, then IR divergent parts are always treated inclusively, so that cancellation of divergences occurs.

IR safe or not?

- Total cross section $\mathcal{O}_n = 1$
- Number of particles $\mathcal{O}_n = n$
- Maximum energy of particle
- Energy flow into particular angular area A of the detector



- Jet cross sections

IR safe or not?

- Total cross section $\mathcal{O}_n = 1$

  ✅

- Number of particles $\mathcal{O}_n = n$

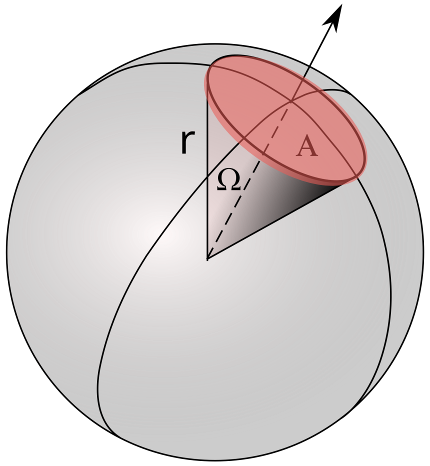
  ❌ soft & collinear unsafe

- Maximum energy of particle

  ❌ collinear unsafe

- Energy flow into particular angular area A of the detector

  ✅



- Jet cross sections ✅ *if* properly defined!

Observables

Collider observables should

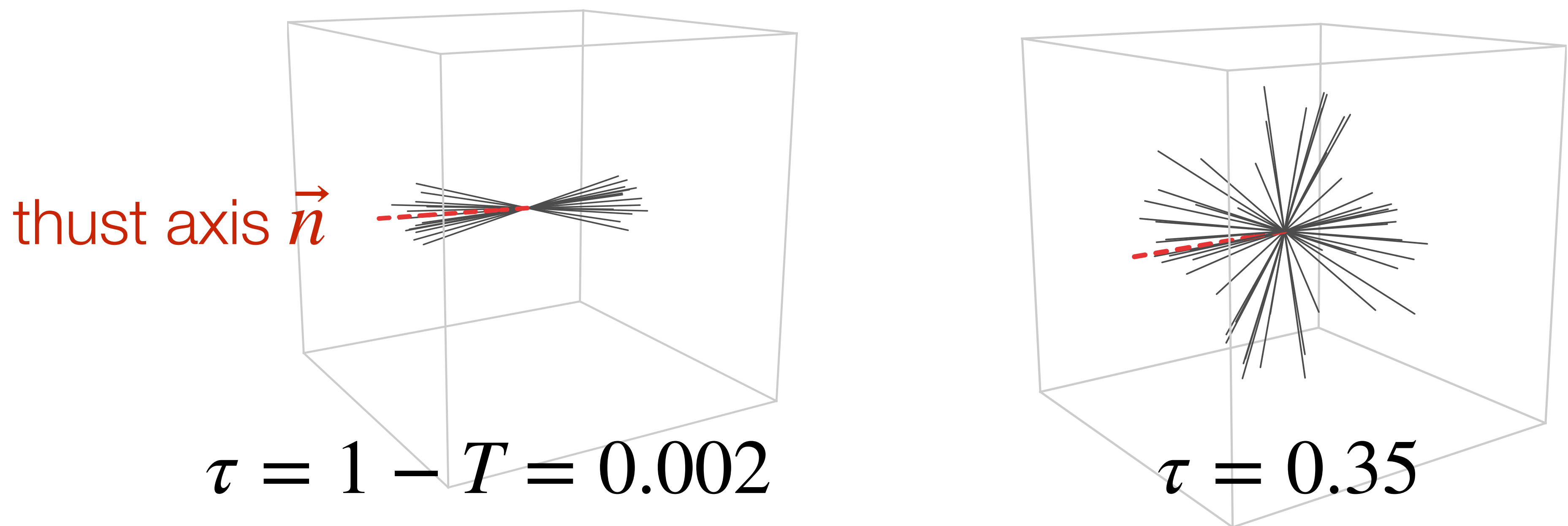
A. not be sensitive to non-perturbative low-energy QCD

B. provide detailed information about short-distance physics

Will now discuss several classes of observables introduced to fulfil these requirements

- Jets, event shapes, energy correlators

Event shapes: e.g. thrust T



Event shape variables parameterize geometric properties of energy and momentum flow.

$$T = \frac{1}{Q} \max_{\vec{n}} \sum_i |\vec{n} \cdot \vec{p}_i|$$

Farhi '77

Generalization to multiple directions and hadronic collisions: *N*-jettiness Stewart, Tackmann, Waalewijn '10

The thrust distribution at LO has the form

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \frac{2\alpha_s}{3\pi}\left[-\frac{3}{\tau}+6+9\tau+\frac{\left(6\tau^2-6\tau+4\right)}{(1-\tau)\tau}\ln\frac{1-2\tau}{\tau}\right]$$

$$= \frac{2\alpha_s}{3\pi}\left[\frac{-4\ln\tau-3}{\tau}+d_{\mathrm{regular}}(\tau)\right]$$

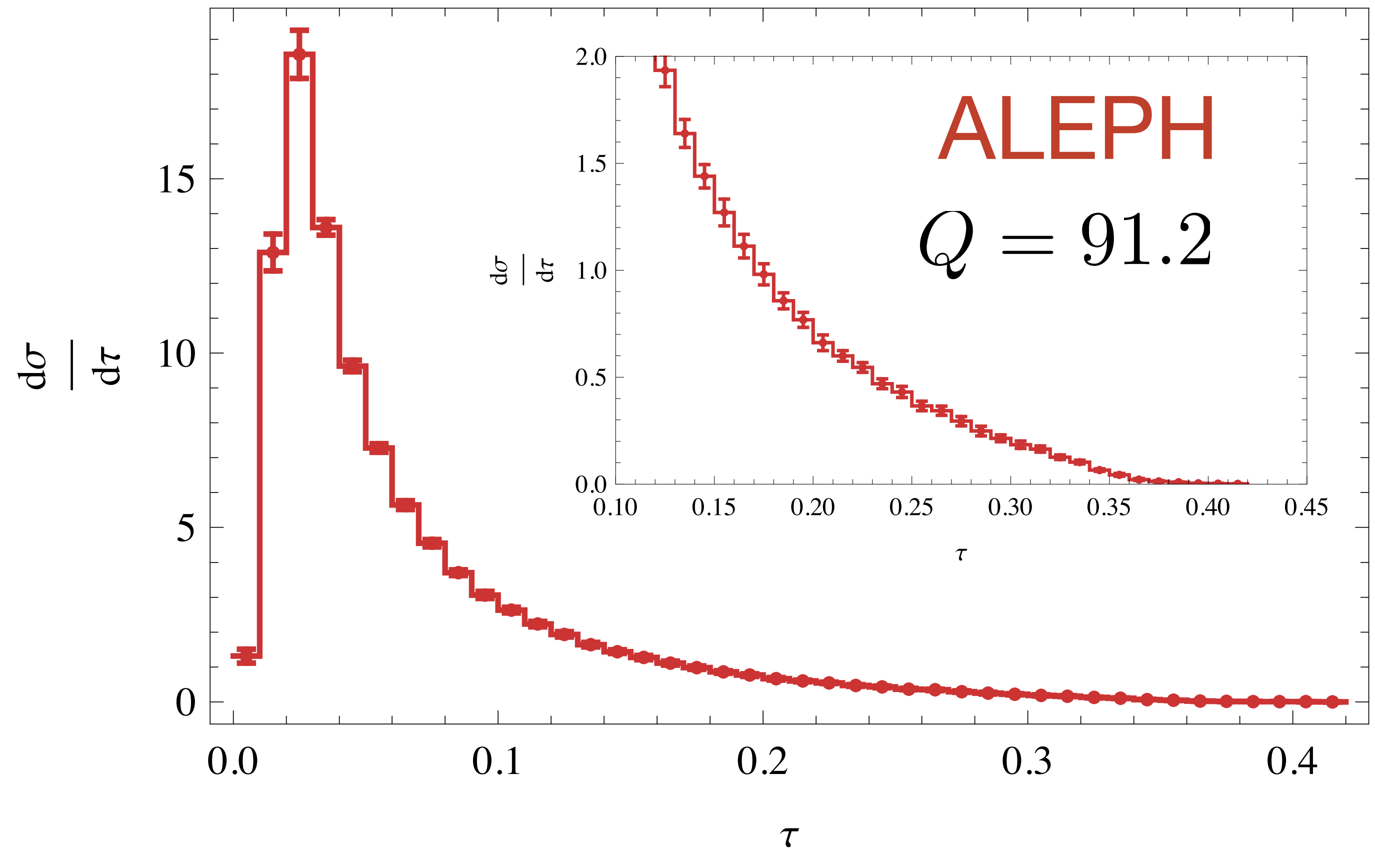
singular terms

Sudakov double logarithm

$$R(\tau) = \int_0^\tau d\tau'\,\frac{1}{\sigma_0}\frac{d\sigma}{d\tau'} = \frac{2\alpha_s}{3\pi}\left[-2\ln^2\tau-3\ln\tau+\ldots\right]$$

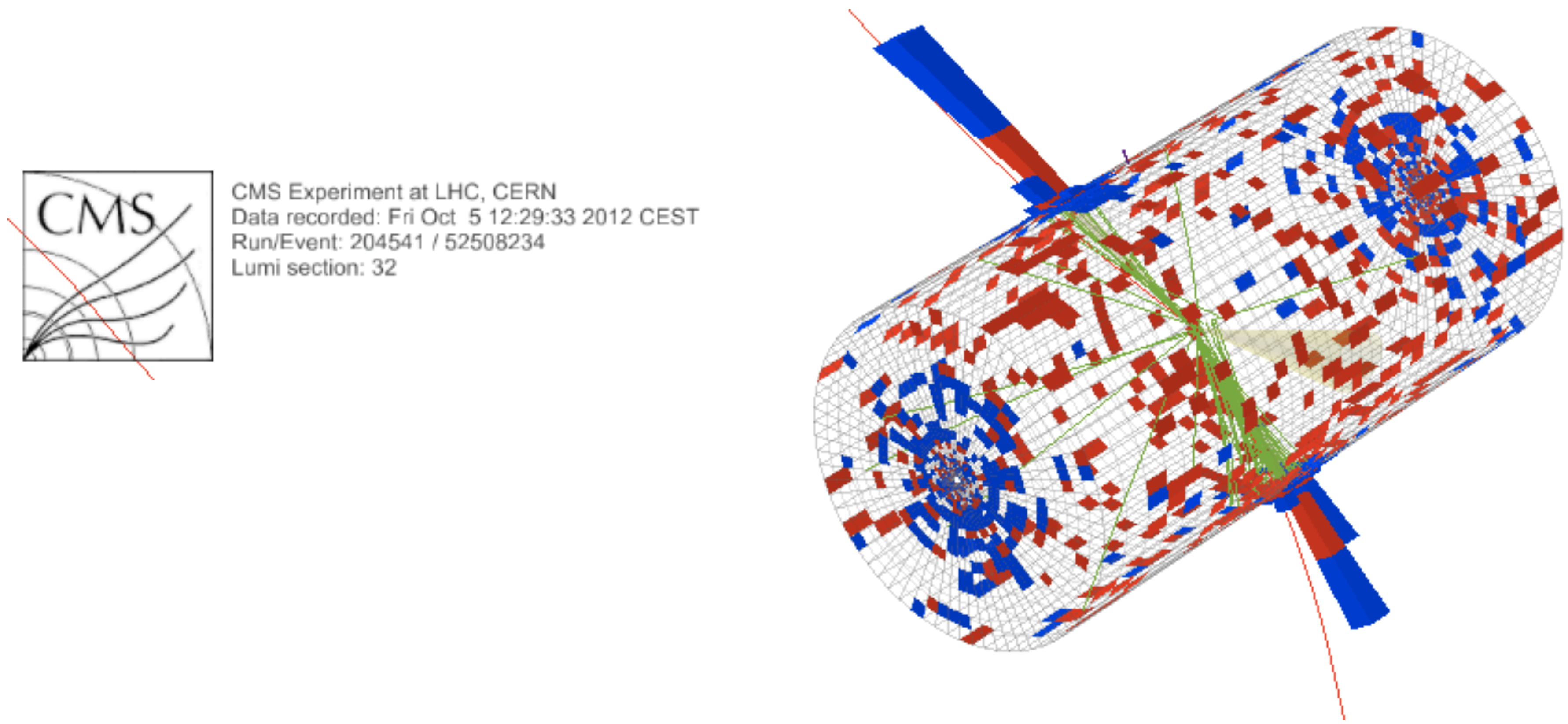
At small τ the perturbative corrections are enhanced!

Fixed-order expansion in α_s breaks down.



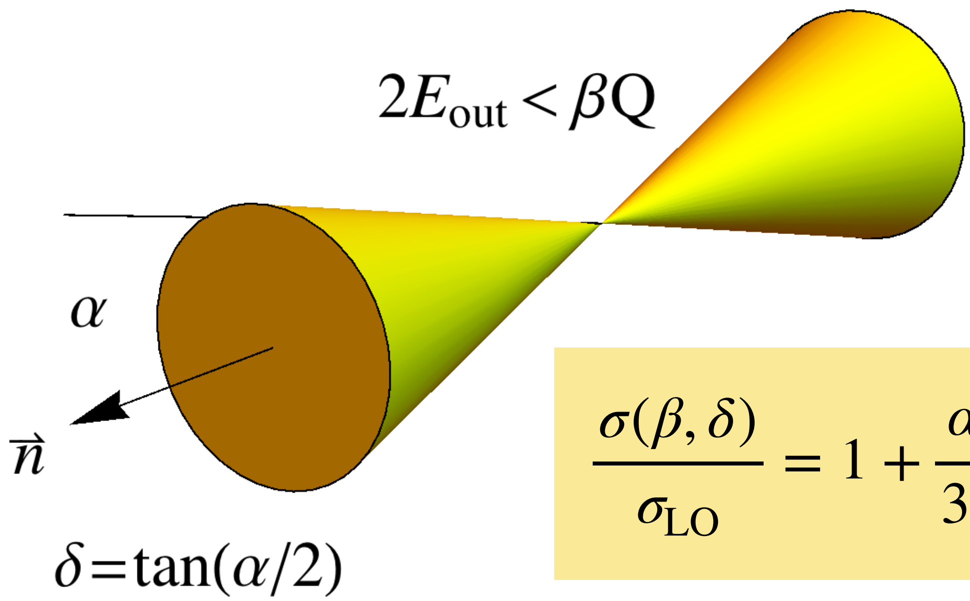
- Precise measurements of thrust and other event shapes at e^+e^- colliders; comparison to theoretical prediction used to extract α_s
- To describe peak region, one needs resummation of logarithmically enhanced terms and include non-perturbative effects
- Sensitivity to soft radiation is problematic at hadron colliders. Solution: Shapes defined with jets, grooming, or soft-insensitive observables such as EEC.

Jet cross sections



Idea: define a cross section which reflects underlying hard partonic process, but includes soft and collinear radiation to be infrared safe.

Sterman-Weinberg ‘77 jets



$$\frac{\sigma(\beta,\delta)}{\sigma_{LO}} = 1 + \frac{\alpha_s}{3\pi}\left(-16\ln\beta\ln\delta - 12\ln\delta + 10 - \frac{4\pi^2}{3}\right) + \mathcal{O}(\beta,\delta)$$

Original definition of a two-jet cross section in e^+e^- collisions. Two parameters

- Cone angle δ, energy fraction β outside cone

Infrared safe, but perturbative corrections are enhanced by $\ln\delta$ and $\ln\beta$. Also, careful analysis shows that lowest scale is $\Lambda = \beta\,\delta\,Q$, must ensure $\Lambda \gg \Lambda_{QCD}$.

Cone jets

To define multijet cone-jet cross sections, one needs IR safe prescriptions to

- choose cone directions
- to treat overlapping cones (split/merge)

Cone algorithms used at the Tevatron relied on seeds and were IR unsafe!

SISCone Salam, Soyez '08 is modern, seedless cone algorithm suited for hadron colliders.

Sequential recombination jets

Alternative definition of a jet is to sequentially combine particles into jets.

Simplest prescription for e^+e^- is JADE algorithm

1. For all pairs of particles ij, compute

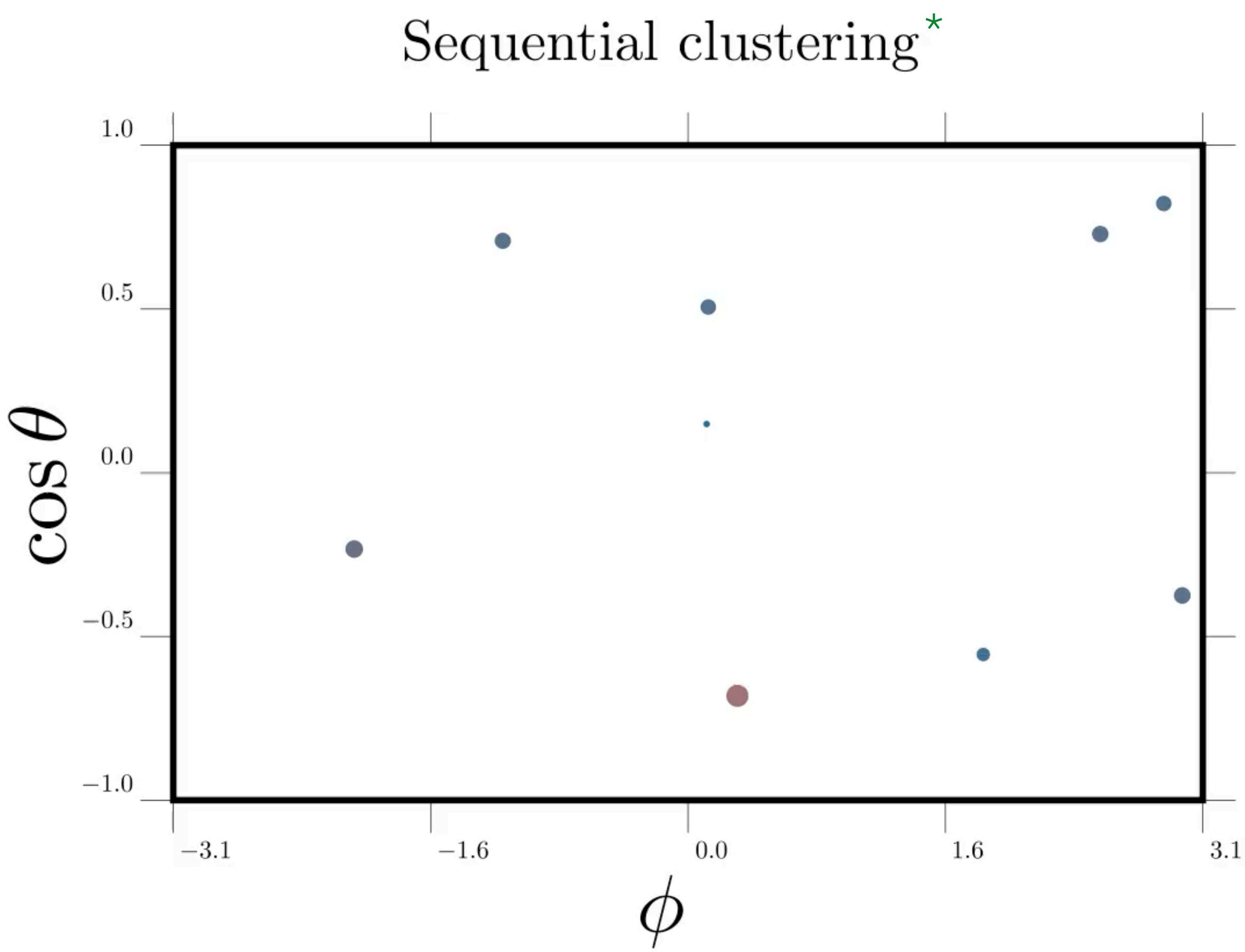
$$d_{ij} = 2E_iE_j(1 - \cos\theta_{ij})/Q^2$$

2. Find pair ij with minimum value $y_{\min} = d_{ij}$.

3. If $y_{\min} < y_{\text{cut}}$ combine pair ij into new particle, go to step 1.)

4. Otherwise declare all remaining particles jets.

Animation by Jürg Haag



1. Find minimum d_{ij}
   $\min(d_{ij}) = ?$
2. Combine i and j into a new jet
3. Stop if $\min(d_{ij})$ is larger than y_{cut}

* clustering is for k_T algorithm (see next slides), not JADE

For massless particles $d_{ij} = (p_i + p_j)^2/Q^2$.

- The JADE algorithms is infrared safe, since soft and collinear particles are immediately combined.

However, jets are quite irregular

- Soft particles moving in opposite directions can end up in same jet
- perturbation theory: $\ln(y_{\rm cut})$ terms with very complicated higher-order structure

k_T algorithm in e^+e^-

Catani et al. '91

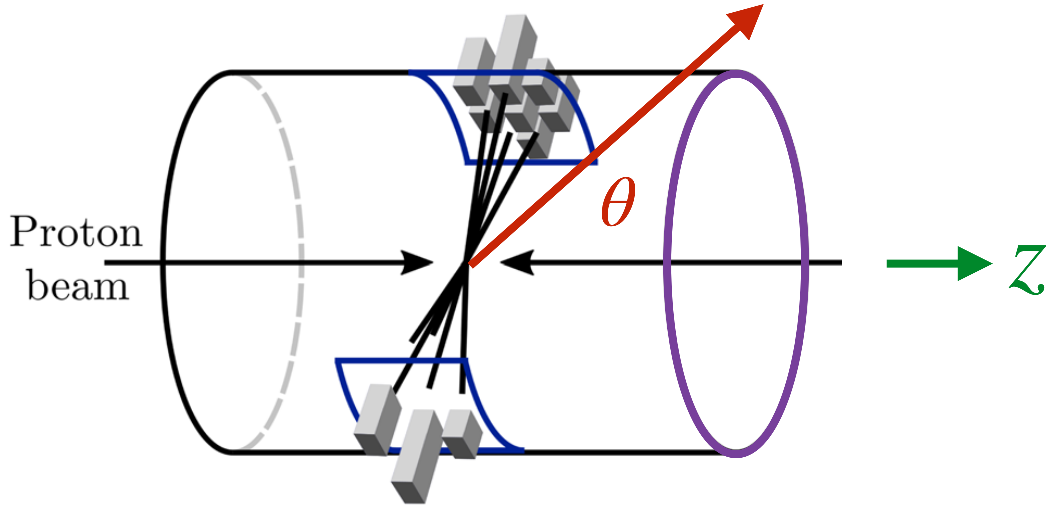
Improved version of the JADE algorithm with distance measure

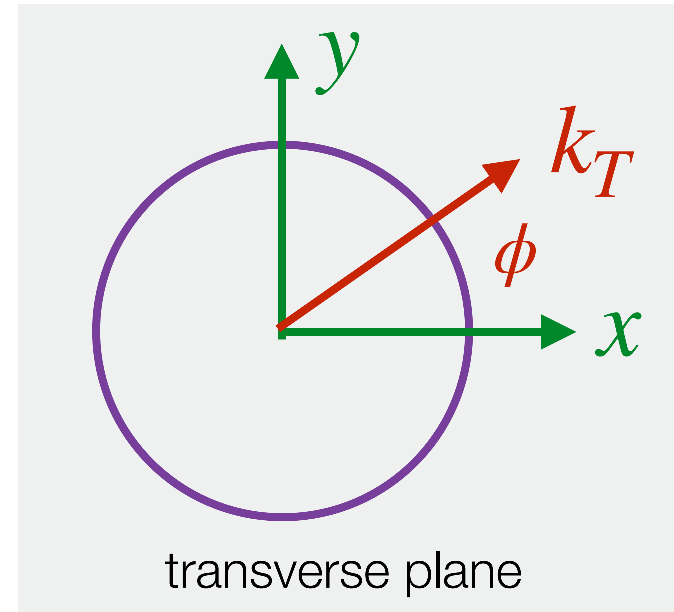
$$d_{ij} = 2\min(E_i^2, E_j^2)\,(1 - \cos\theta_{ij})/Q^2$$

Modification

- ensures that soft partons are clustered with nearby partons
- if i is softer parton then $d_{ij} \approx E_i^2\theta_{ij}^2$,
  transverse momentum of i relative to particle j

Hadron collider kinematics





Partons (quarks and gluons) of the protons collide with different energies. Lab frame ≠ partonic center of mass frame. Use variables invariant under boosts along beam axis:

- Momentum transverse to the beam k_T, azimuthal angle ϕ and rapidity differences Δy

Rapidity

$$y = \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right)$$

Pseudo-rapidity

$$\eta = \frac{1}{2}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$$

massless particles:

$$y = \eta$$

k_T algorithm for hadron colliders

Catani et al. '93; Ellis and Soper '93

Distance measure

$$d_{ij} = \min(k_{Ti}^{2p}, k_{Tj}^{2p}) \frac{\Delta R_{ij}^2}{R^2}$$

$$d_{iB} = k_{Ti}^{2p}$$ distance to beam

with angular distance

$$\Delta R_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2$$

- Parameter R is the ``jet radius"
- k_T algorithm: $p = 1$; C/A algorithm: $p = 0$

cluster soft into nearby

"Cambridge/Aachen" cluster solely based on angle

Clustering sequence for hadron collider algorithms

1. Compute beam distance d_{iB} and distance d_{ij} for all pairs
2. If mininum is d_{ij} then recombine, go to step 1
3. If mininum is d_{iB} then i declare i a jet and remove it from list

Inclusive algorithm: all particles are clustered into jets and many jets have very low k_T

- Hard jets selected by imposing minimum $k_T^{\min}$ on jets
- Exclusive n-jet samples by vetoing additional jets above $k_T^{\min}$

anti-k_T algorithm

Cacciari, Salam, Soyez '08; > 11'000 citations

Experimentally, k_T and $\mathrm{C/A}$ jets were unpopular, because the jets had irregular shapes.

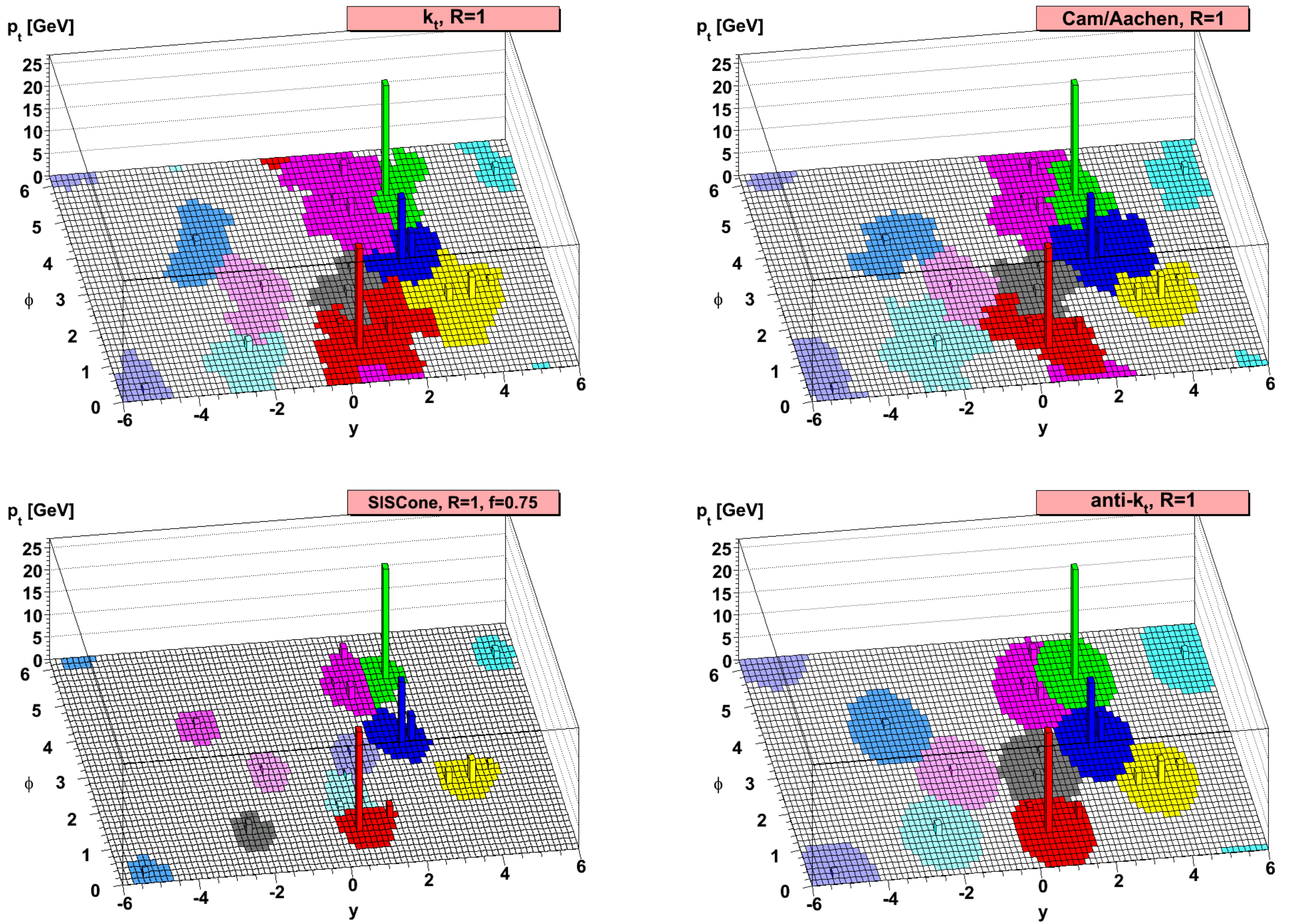
Problem is solved by setting $p = -1$ (anti-k_T algorithm)

$$d_{ij} = \min(k_{Ti}^{-2}, k_{Tj}^{-2})\frac{\Delta R_{ij}^2}{R^2} \quad \text{and} \quad d_{iB} = k_{Ti}^{-2}$$

Reverses clustering sequence:

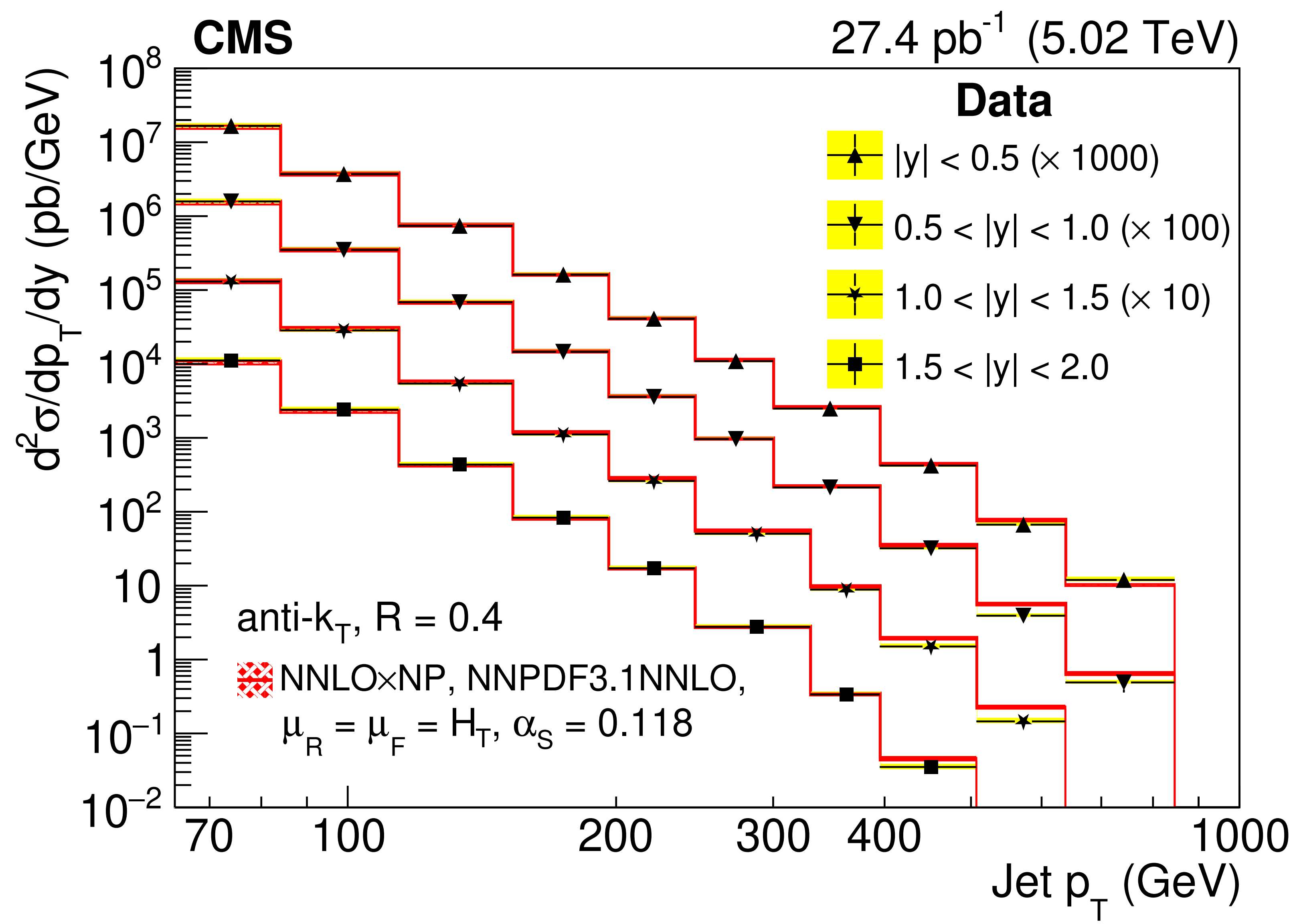
- start with hardest parton, cluster nearby softer particles into it

and leads to very cone-like jets! Default LHC algorithm.



parton shower event + many additional very soft partons Cacciari, Salam, Soyez '08

Inclusive jet cross section $pp \to \text{jet} + X$



- Plot shows p_T and rapidity y of leading jet.
- NNLO theory prediction needs PDFs (see next lecture), nonperturbative (NP) effects estimated by parton shower

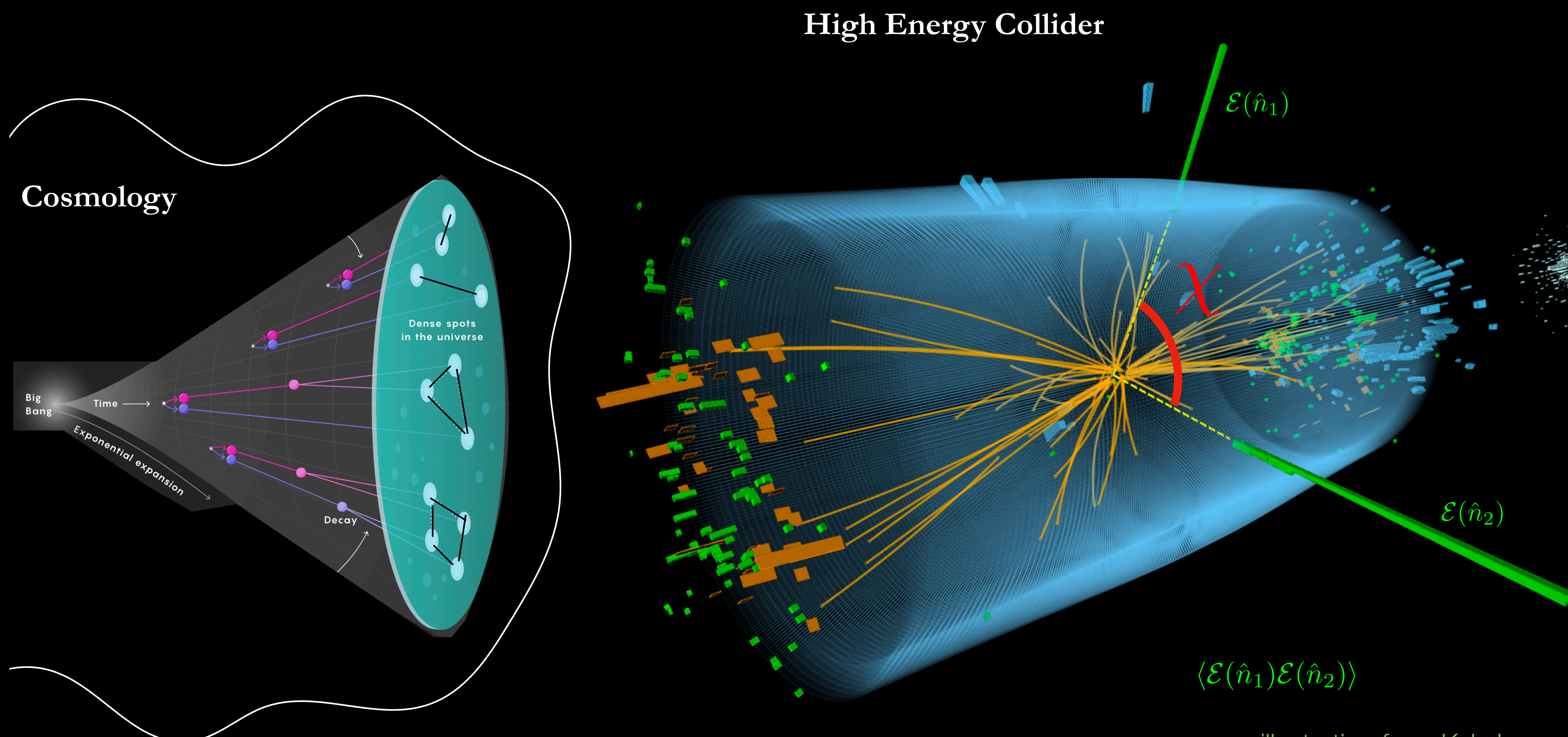
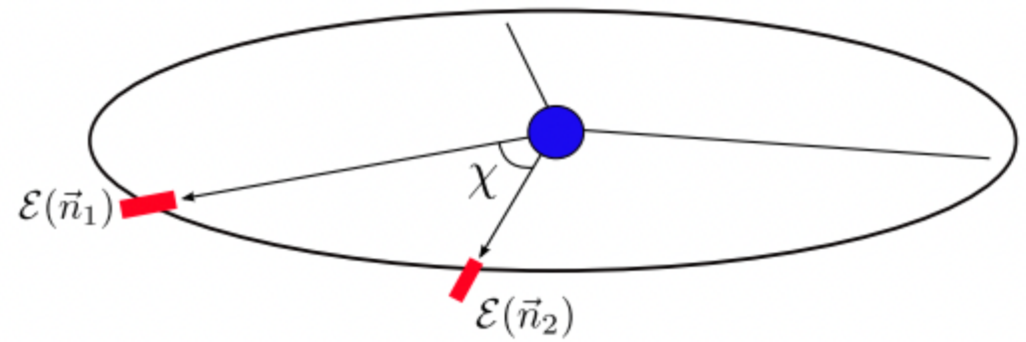


illustration from Kyle Lee

Energy-Energy Correlators (EECs)



Matrix elements

Energy-flow operator

$$\langle\Psi|\mathcal{E}(\hat{n}_1)\cdots\mathcal{E}(\hat{n}_k)|\Psi\rangle \quad \text{with} \quad \mathcal{E}(\hat{n}) = \int_0^\infty dt \lim_{r\to\infty} r^2 n^i T_{0i}(t, r\hat{n})$$

Sveshnikov, Tkachov ‘95

characterize energy flow into the detector

A lot of new interesting developments in using these energy-energy correlators to study jet subtructure, determine α_s and m_t, …

Correlators have many good properties

- weighted by energy: insensitive to soft radiation:
- factorization, light-ray OPE, CFT techniques Hofman, Maldacena ‘08

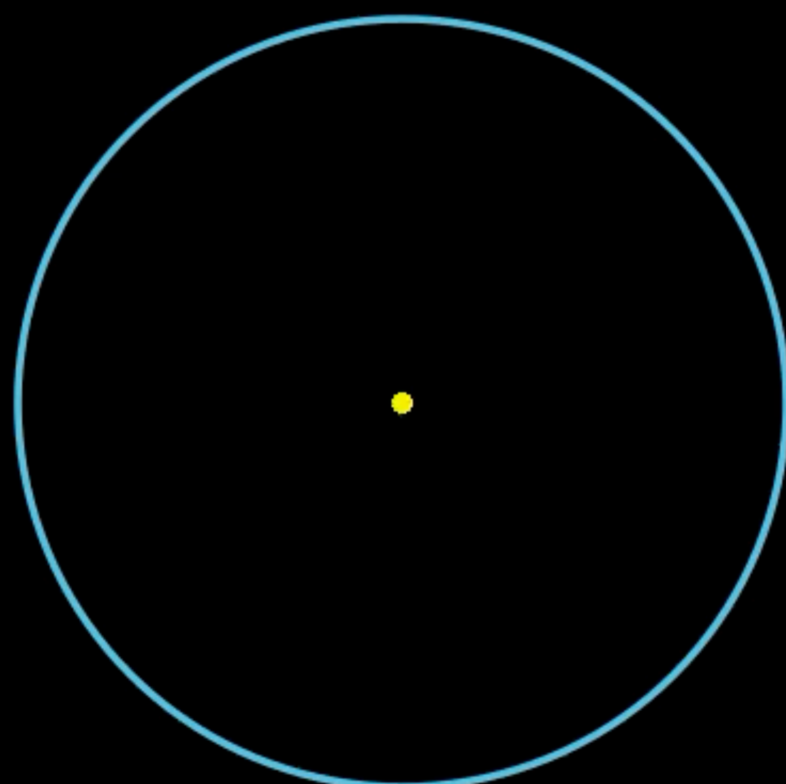
2-point EEC

Simplest correlator is two-point function

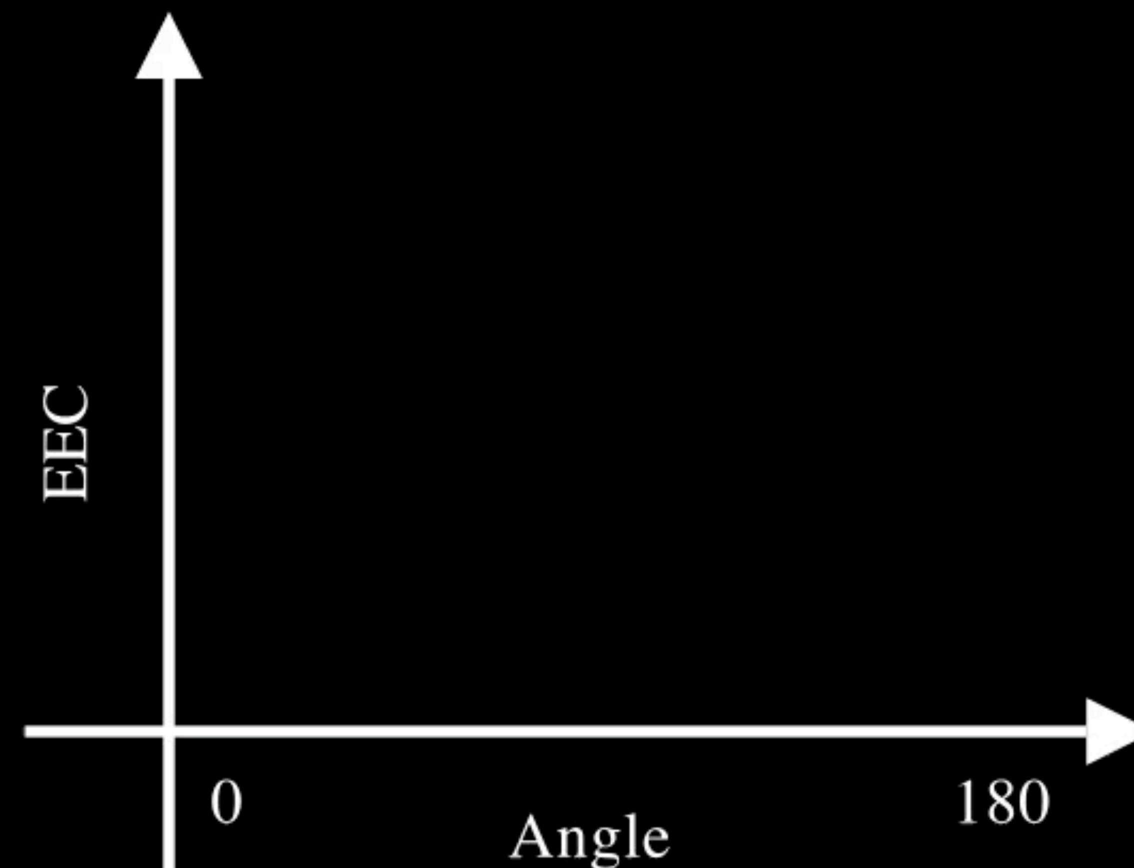
$$\mathrm{EEC}(\chi) = \sum_{a,b} \int \mathrm{d}\sigma_{e^+e^- \to a+b+X} \, \frac{E_a E_b}{Q^2} \, \delta(\cos\chi_{ab} - \cos\chi)$$

Basham, Brown, Ellis, Love '78

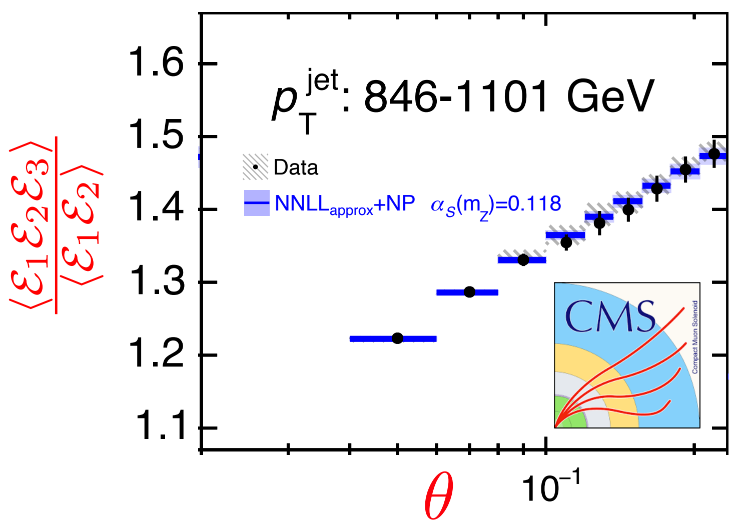
- Record intermediate angle χ between pairs of particles a and b, and put product $E_a E_b$ of their energies into χ histogram.
- In contrast to event-shapes and jets, each event has multiple entries into histogram!

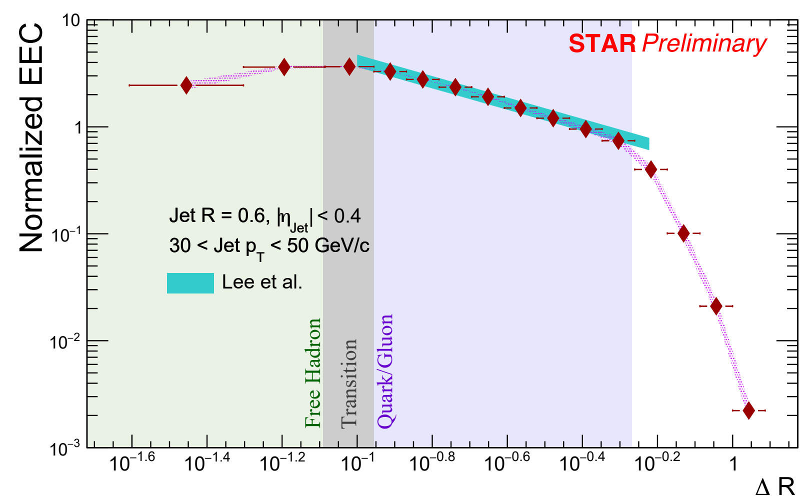


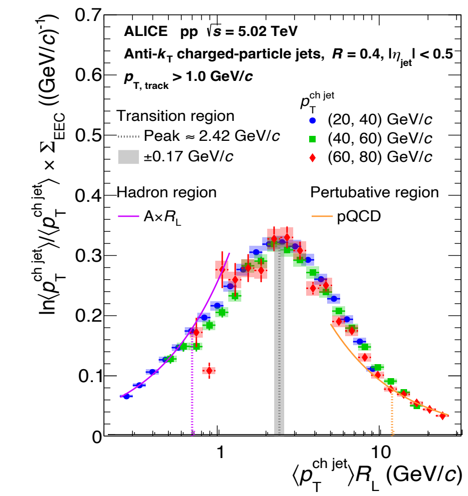
Credit: Hua Xing Zhu



EEC measurements at LHC





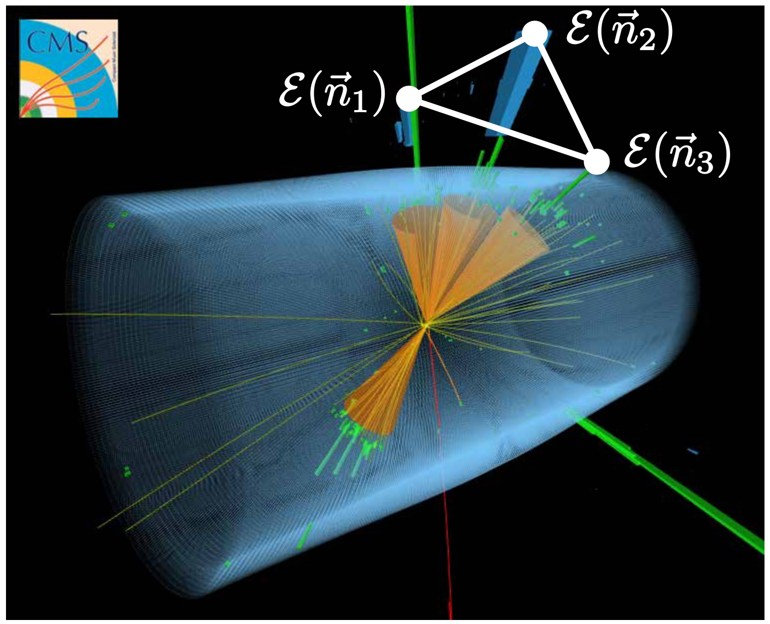


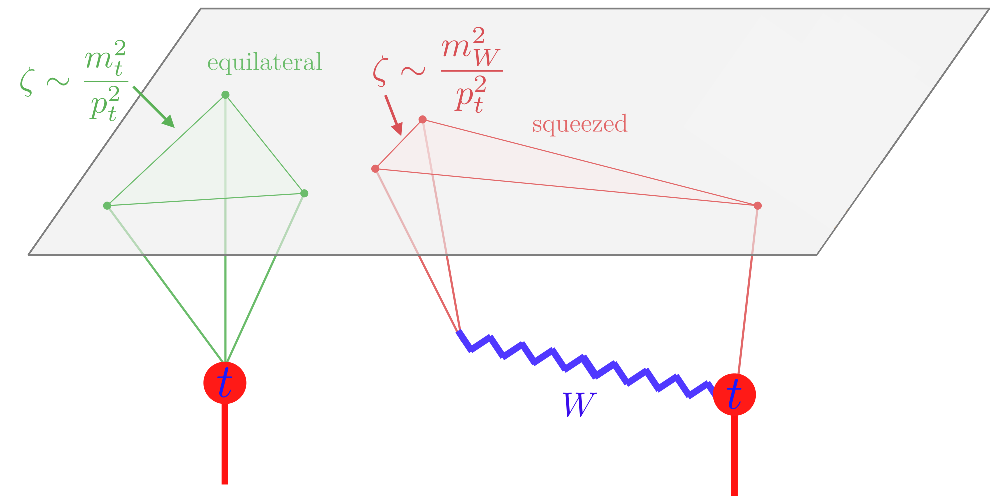
NLL: Lee, Mecaj, Moult '22

- Now many measurements of transverse EECs within jets at hadron colliders (ALICE, ATLAS, CMS, STAR).
- Large angles: perturbative. Very small angles: hadronisation
- CMS α_s determination based prediction with resummation Chen, Gao, Li, Xu, Zhang, Zhu '23

$$\alpha_s(M_Z) = 0.1229^{+0.0014}_{-0.0012}\,(\text{stat})^{+0.0030}_{-0.0033}\,(\text{theo})^{+0.0023}_{-0.0036}(\text{exp})$$

Jet substructure from EECs





Top quark jets have substructure from top decay

$$t \to W^+ + b \quad \text{and} \quad W^- \to u\bar{d}$$

Proposals to extract ratio m_t/m_W from 3-point correlator in top decays Holguin, Moult, Pathak, Procura, Schöfbeck, Schwarz ’23, ’24; Xiao, Ye, Zhu ‘24

Part III

Factorization, evolution, resummation

- Soft and collinear factorization
- Parton distribution functions
- DGLAP evolution
- Drell-Yan process

QCD made simple(r)

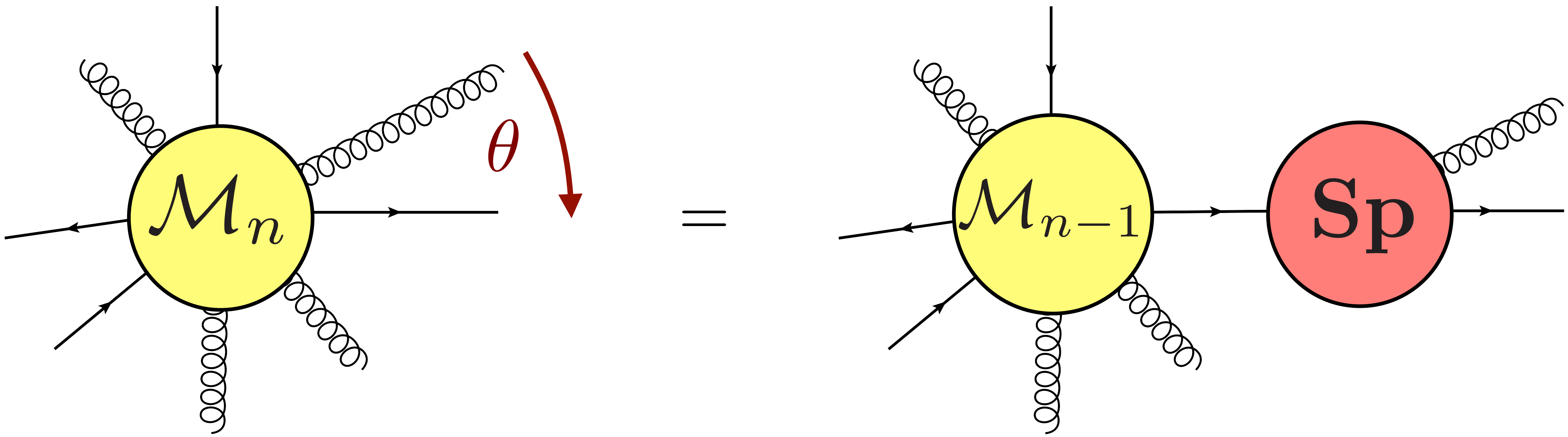
The perturbative expressions for the scattering of quarks and gluons simplify considerably in the

- Collinear limit, where multiple particles move in a similar direction.
- Soft limit, in which particles with small energy and momentum are emitted.

Cross sections are enhanced

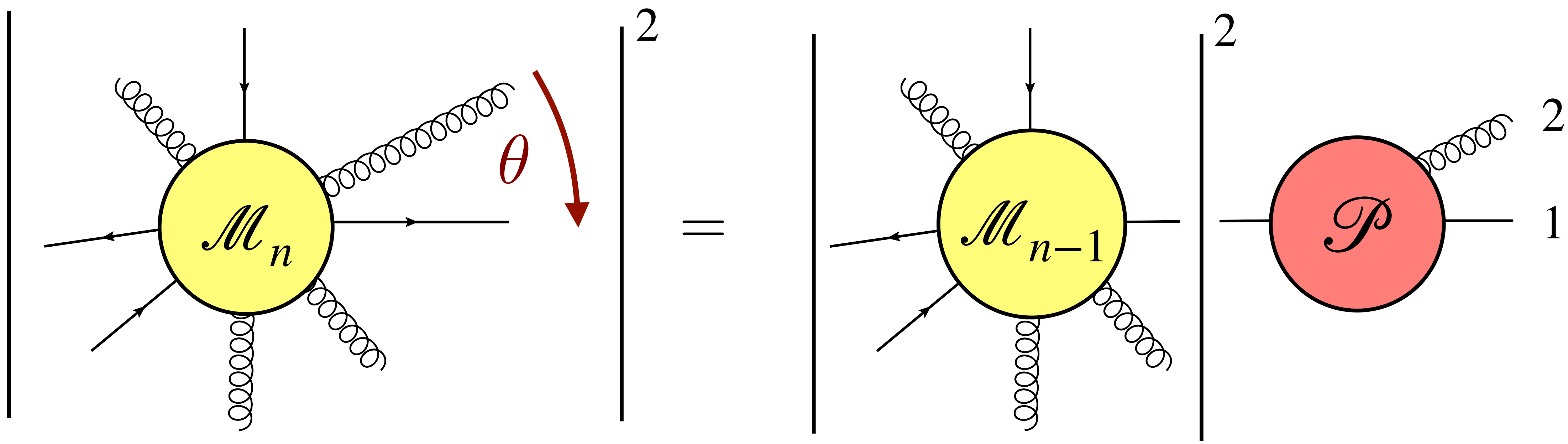
- IR singularities cancel for IR safe observables, but
- induce large logarithms (see e.g. thrust, SW-jets) which should be resummed to all orders.

Collinear limit



In the limit $\theta \to 0$, where the partons become collinear, the n-parton amplitude factorizes into a product of an $(n-1)$-parton amplitude times a splitting amplitude $\mathbf{Sp}$.

Collinear limit



Factorization is particularly simple, if we square the amplitude and sum over spins

splitting functions

$$|\mathcal{M}_n(p_1, p_2, \ldots, p_n)|^2 = \frac{g_s^2}{p_1 \cdot p_2} \mathscr{P}_{P\to 1+2}(z) |\mathcal{M}_{n-1}(P, \ldots, p_n)|^2$$

Collinear kinematics: $p_1 \approx z\,P$ and $p_2 \approx (1-z)\,P$
with momentum fraction $0 < z < 1$

The splitting functions

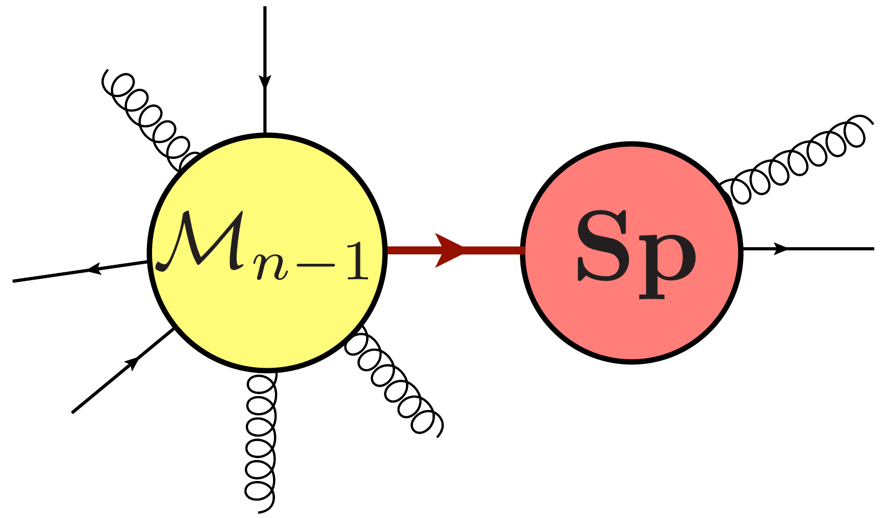
$$\mathscr{P}_{q\to q+g}(z) = C_F \left[\frac{1+z^2}{1-z}\right]$$

$$\mathscr{P}_{g\to \bar{q}+q}(z) = T_F \left[1 - 2z(1-z)\right]$$

$$\mathscr{P}_{g\to g+g}(z) = 2C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right]$$

play an important role in QCD, e.g. in PDF evolution and in parton showers (next lecture). Short-hand notation

$$\mathscr{P}_{a\to b}(z) \quad \text{for} \quad \mathscr{P}_{a\to b+c}(z)$$



The splitting amplitude diverges as $\theta \to 0$ and the factorization holds up to regular terms

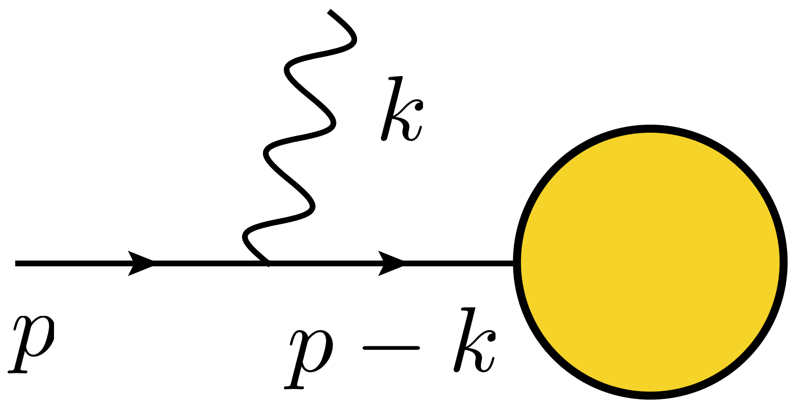
For the cross section, one finds

$$\mathrm{d}\sigma_n \sim d\sigma_{n-1} \frac{\mathrm{d}\theta}{\theta} \frac{dE_g}{E_g} d\phi$$

Logarithmic enhancements at small angle, and also at small gluon energy. No interference!

Soft limit

Also when particles with small energy and momentum are emitted, the amplitudes simplify:



$$\cdots \frac{\not p - \not k + m}{(p-k)^2 - m^2} \gamma_\mu \, u(p)$$

$$\approx \; \ldots \; u(p) \frac{p_\mu}{p \cdot k}$$

Soft emission factors from the rest of the amplitude.

$p \cdot k = E\,\omega\,(1 - \cos\theta)$ in denominator leads to logarithmic enhancements at small energy and small angle.

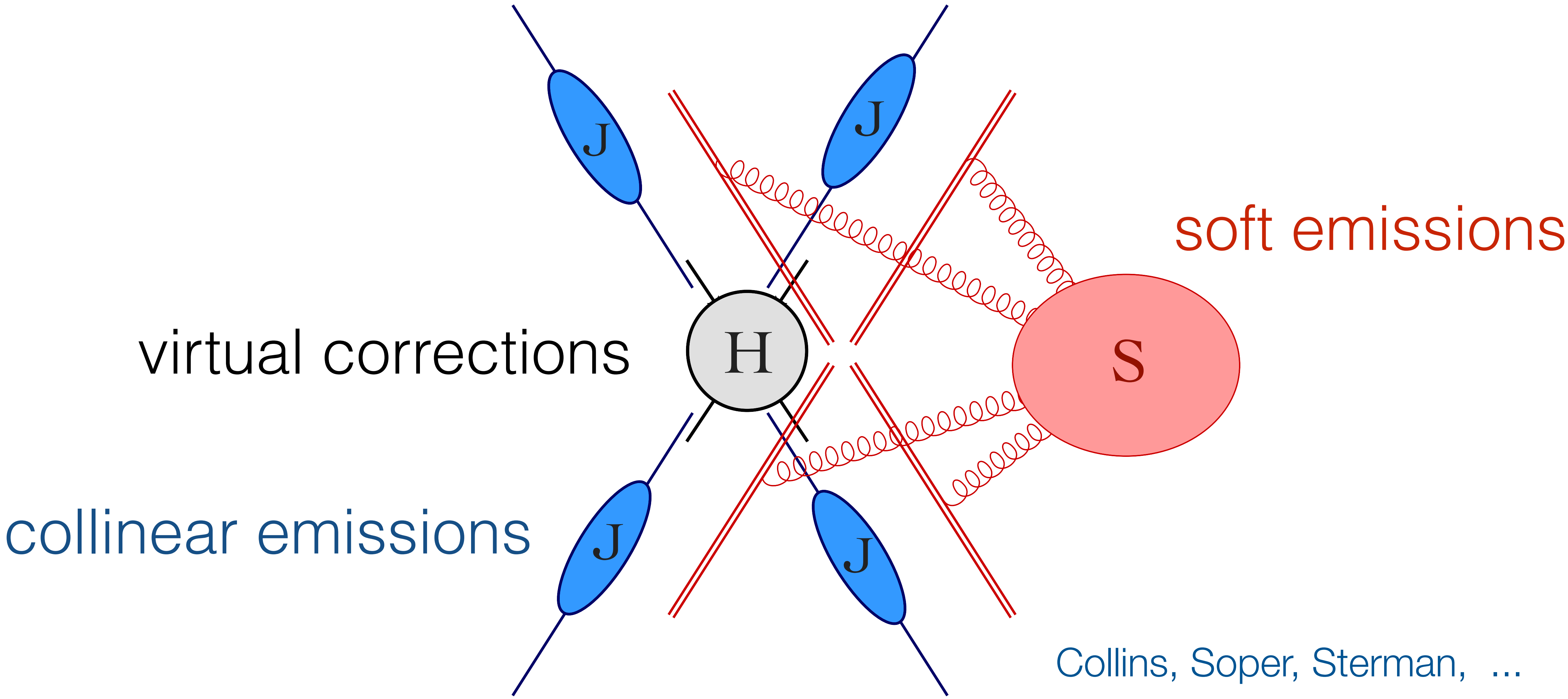
The cross section for the emission of one gluon is

$$d\sigma_{n+1}^{\mathrm{soft}} = \frac{\alpha_s}{2\pi}\,\frac{d\omega}{\omega}\,\frac{d\Omega}{2\pi}\,\sigma_n \sum_{i,j=1}^{n} C_{ij}\,\frac{\omega^2\, p_i \cdot p_j}{p_i \cdot k\, p_j \cdot k}$$

color factor ~ $\boldsymbol{T}_i \cdot \boldsymbol{T}_j$

So for massless particles soft emission is a *pure interference effect*, in marked contrast to collinear emissions!

Soft-collinear factorization



Collins, Soper, Sterman, ...

Basis for higher-log resummation. More complicated than structure than what's implemented in a parton shower:

- Interference, color structure, spin, loop corrections.

Soft-Collinear Effective Theory (SCET)

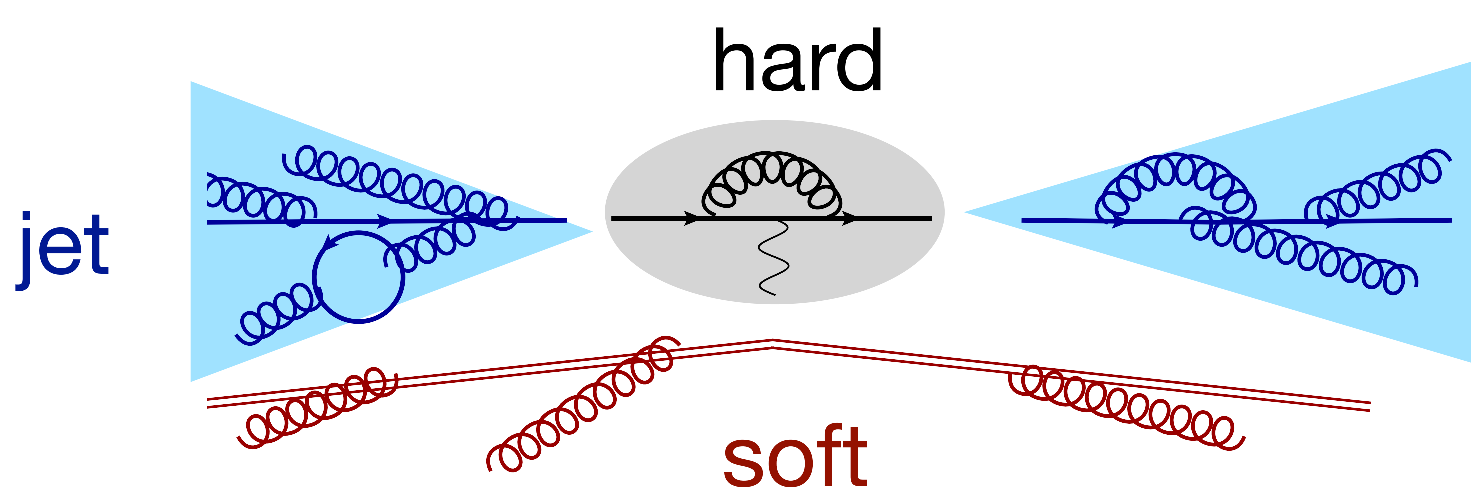
Bauer, Pirjol, Stewart et al. 2001, 2002; Beneke, Diehl et al. 2002; ...

Implements interplay between soft and energetic collinear particles into effective field theory

Hard } high-energy

Collinear *fields* } low-energy part

Soft *fields*



Allows one to analyze **factorization** of cross sections and perform **resummations** of large Sudakov logarithms.

Diagrammatic Factorization

The simple structure of soft and collinear emissions forms the basis of the classic factorization proofs, which were obtained by analyzing Feynman diagrams.

Collins, Soper, Sterman 80's ...

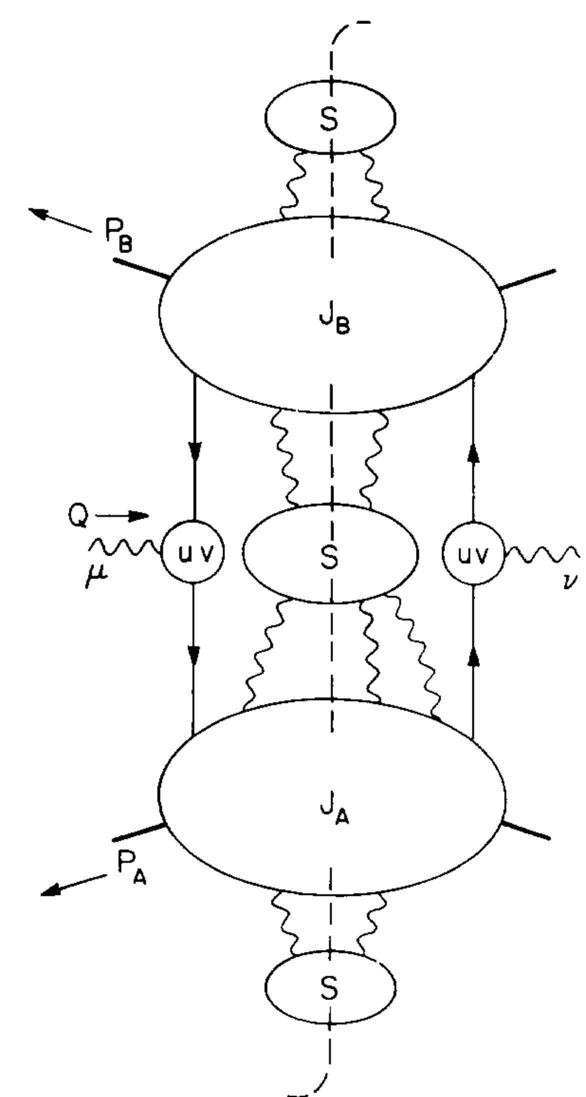
Advantages of the the SCET approach:

Simpler to exploit gauge invariance on the Lagrangian level

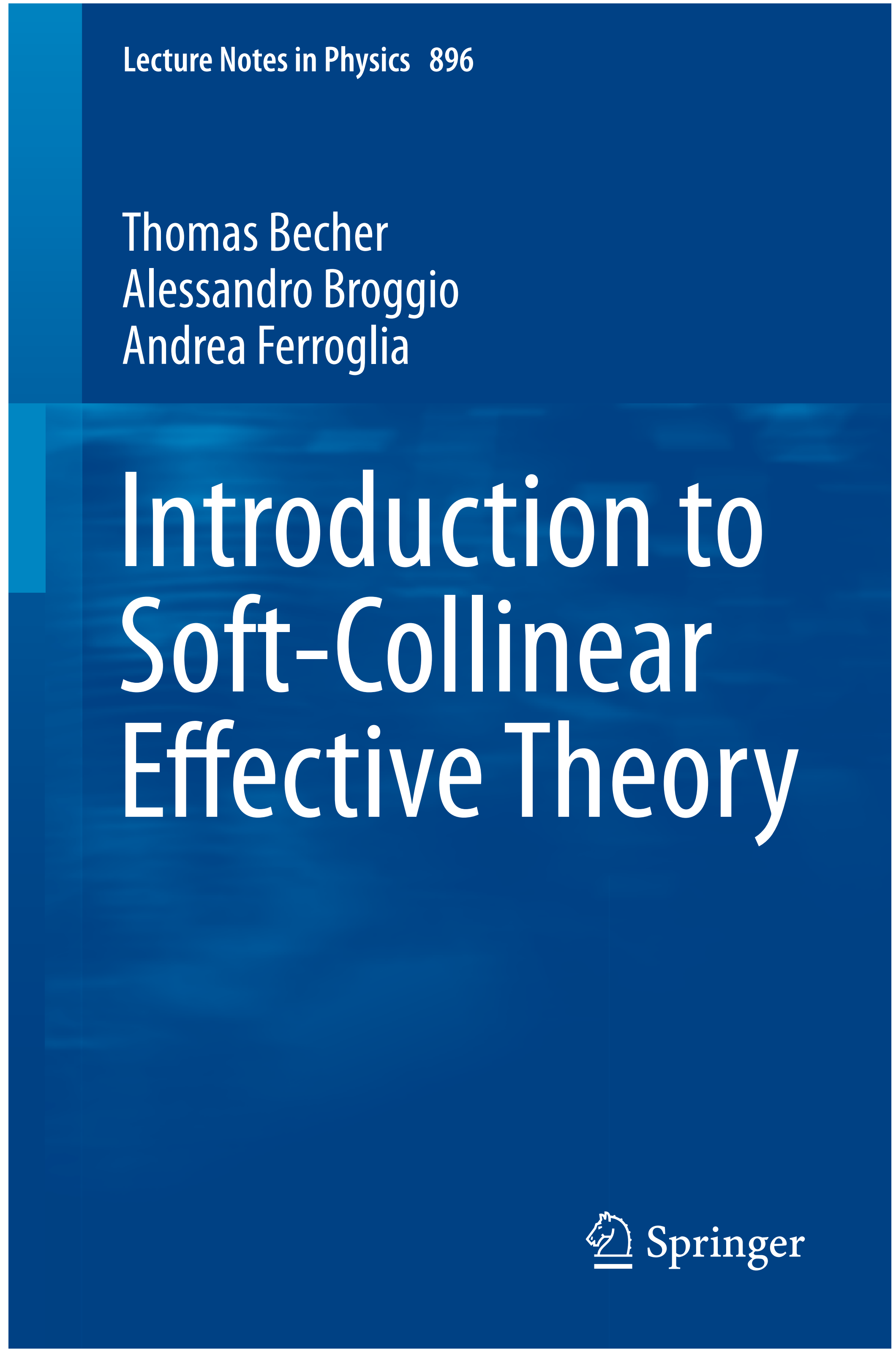
Operator definitions for the soft and collinear contributions

Resummation with renormalization group

Can include power corrections



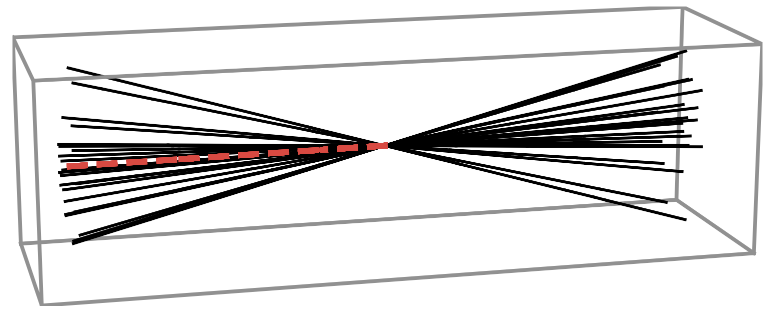
Collins and Soper '81



arXiv:1410.1892

Example: factorization for Thrust

$$T = \max_{\mathbf{n}} \frac{\sum_i |\mathbf{p}_i \cdot \mathbf{n}|}{\sum_i |\mathbf{p}_i|}$$



$$1 - T \approx \frac{M_1^2 + M_2^2}{Q^2}$$

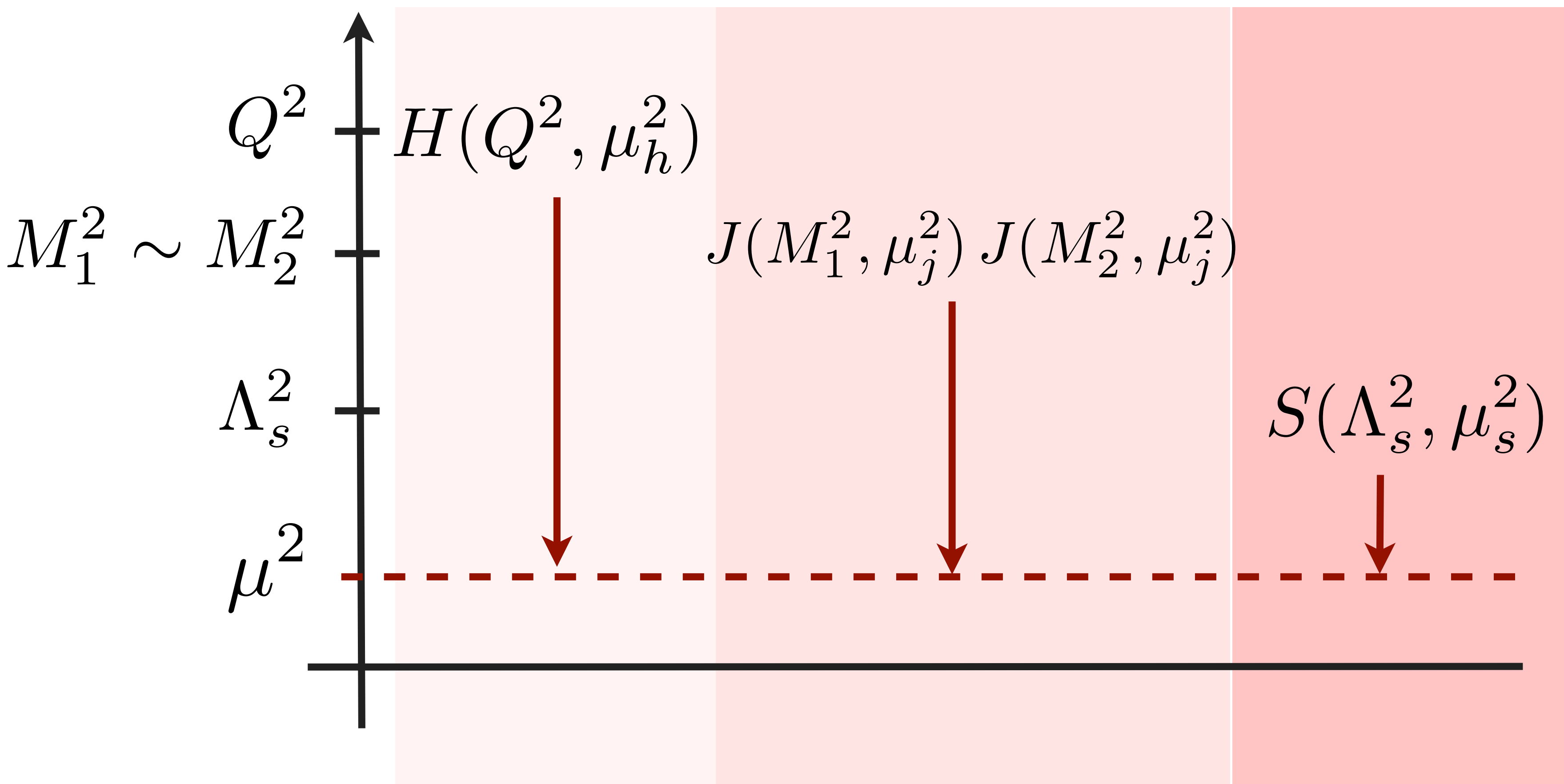
- The perturbative result for the thrust distribution contains logarithms $\alpha_s^2 \ln^{2n}(\tau)$, where $\tau = 1 - T$.
    - Near the end-point $\tau \to 0$ the logarithmic terms dominate.
- Using SCET one can derive a factorization theorem

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = H(Q^2,\mu) \int dM_1^2 \int dM_2^2 \, J(M_1^2,\mu)\, J(M_2^2,\mu)\, S_T\!\left(\tau\, Q - \frac{M_1^2 + M_2^2}{Q}, \mu\right)$$

Scales:	Q^2	$\gg$	$M^2 \sim \tau Q^2$	$\gg$	τQ
	hard		collinear		soft

Resummation by RG evolution

Evaluate each part at its characteristic scale, evolve to common reference scale μ



Each contribution is evaluated at its natural scale. No large perturbative logarithms.

RG-improved perturbation theory

Aside: counting of logarithms

The integrated cross section

$$\Sigma(\tau) = \frac{1}{\sigma}\int_0^{\tau} d\tau' \frac{d\sigma}{d\tau'}$$

has for low q_T an expansion of the form ($L = \ln \tau$)

$$\Sigma(\tau) = 1 + \alpha_s \left(c_2 L^2 + c_1 L + c_0\right) + \alpha_s^2 \left(c_4 L^4 + c_3 L^3 + \ldots\right) + \alpha_s^3 \left(c_6 L^6 + \ldots\right) + \ldots$$

leading logarithms

next-to-leading logarithms

Exponentiation

The resummed cross section has the form

$$\Sigma(\tau) = \exp\left(L\, g_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \alpha_s^2 g_4(\alpha_s L) + \dots\right)$$

Nontrivial, crucial feature: only one *L* per order

Accuracy:

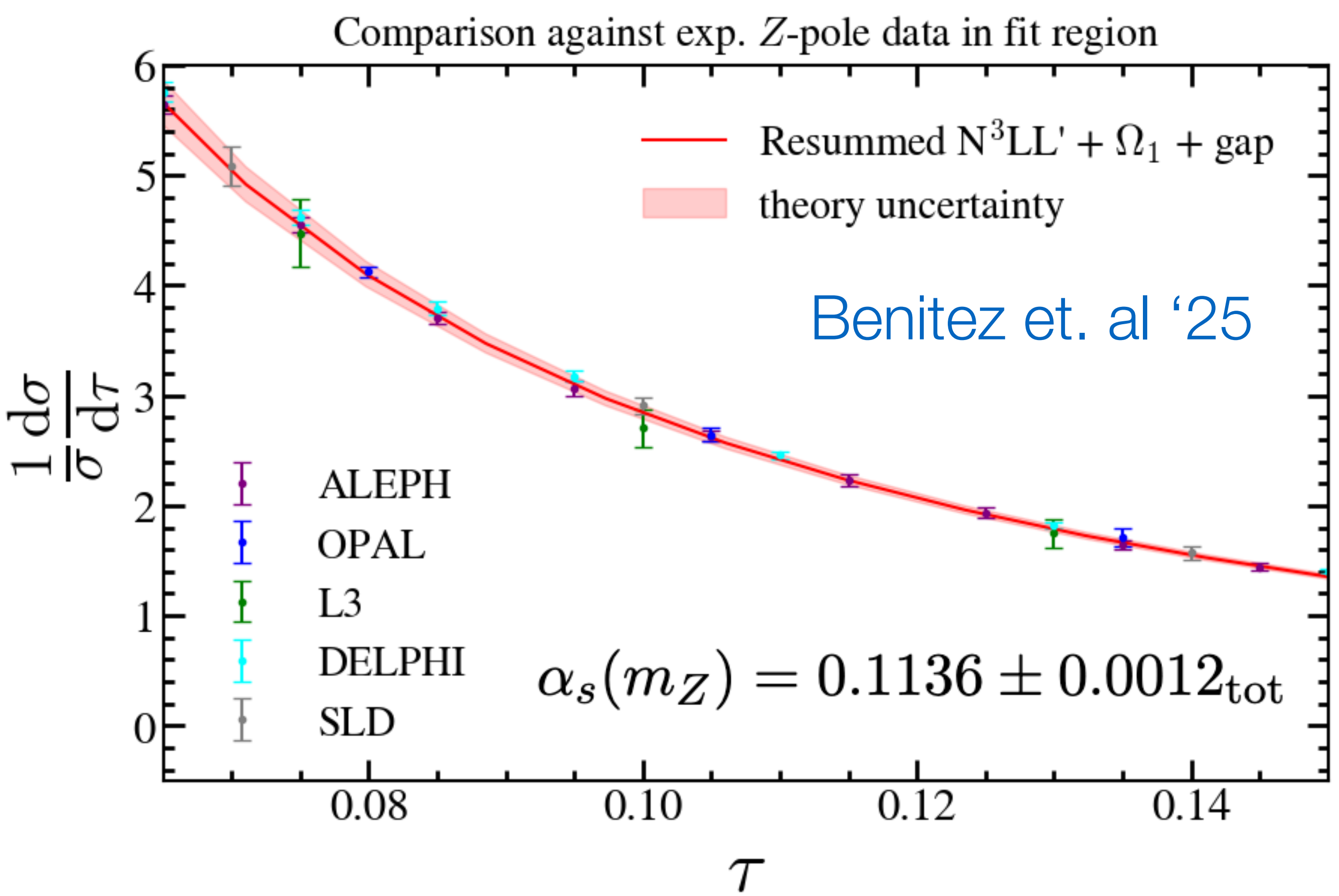
- LL: g_1; NLL: g_1, g_2; NNLL: g_1, g_2, g_3

Systematics: expand in α_s but count $\alpha_s L$ as $O(1)$

Matching:

N³LL + NNLO

logarithms at small τ + fixed order at larger τ



State of the art theoretical predictions for thrust include

- NNLO fixed order + resummation up to N^3LL + fit for hadronisation effects (parameter Ω_1)

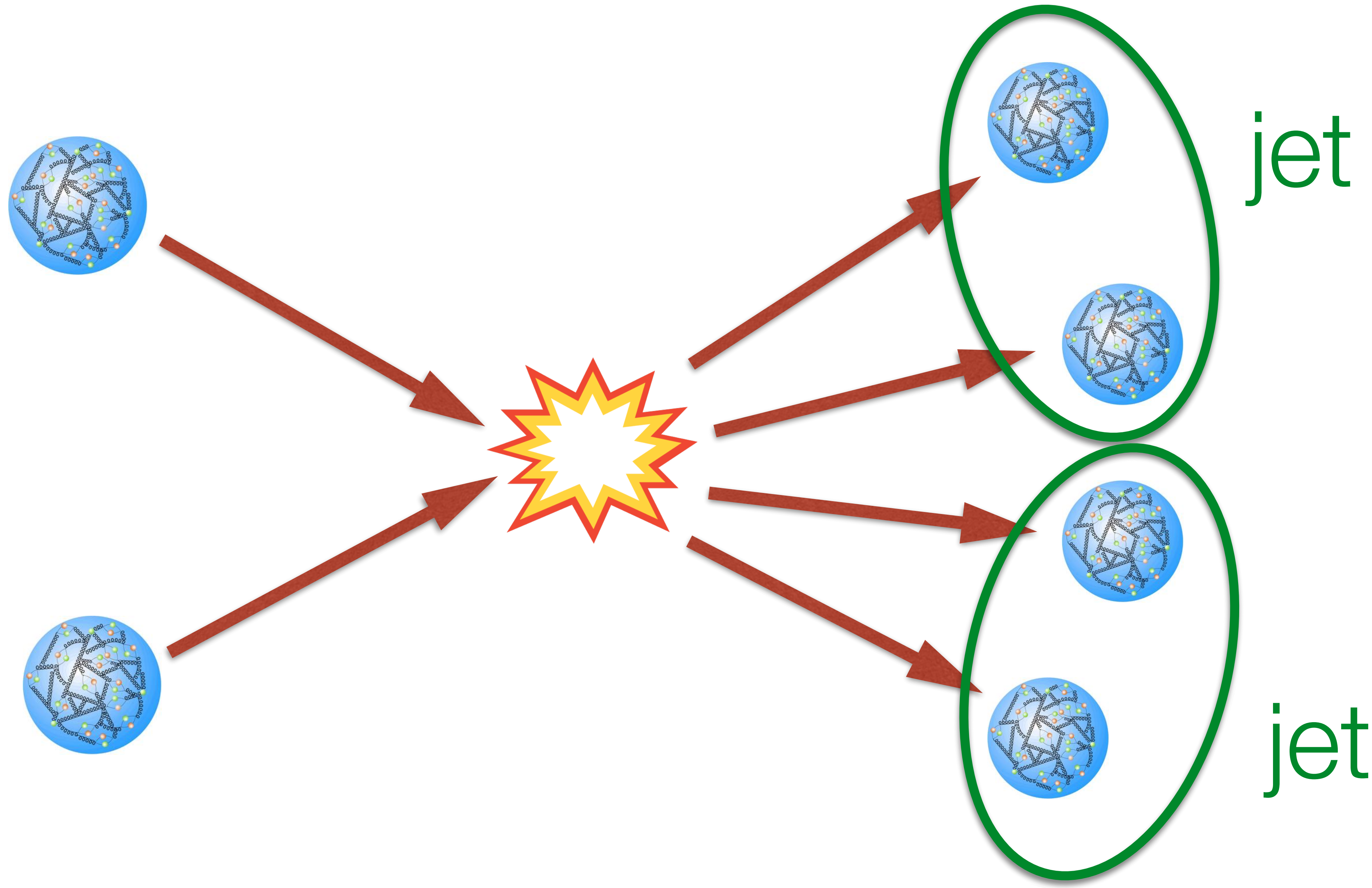
- Fit to data gives low α_s in strong tension with world average

Ongoing discussions (see e.g. talks by Benitez, Ferrera and Nason at PSR2025 conference) how this can be resolved.

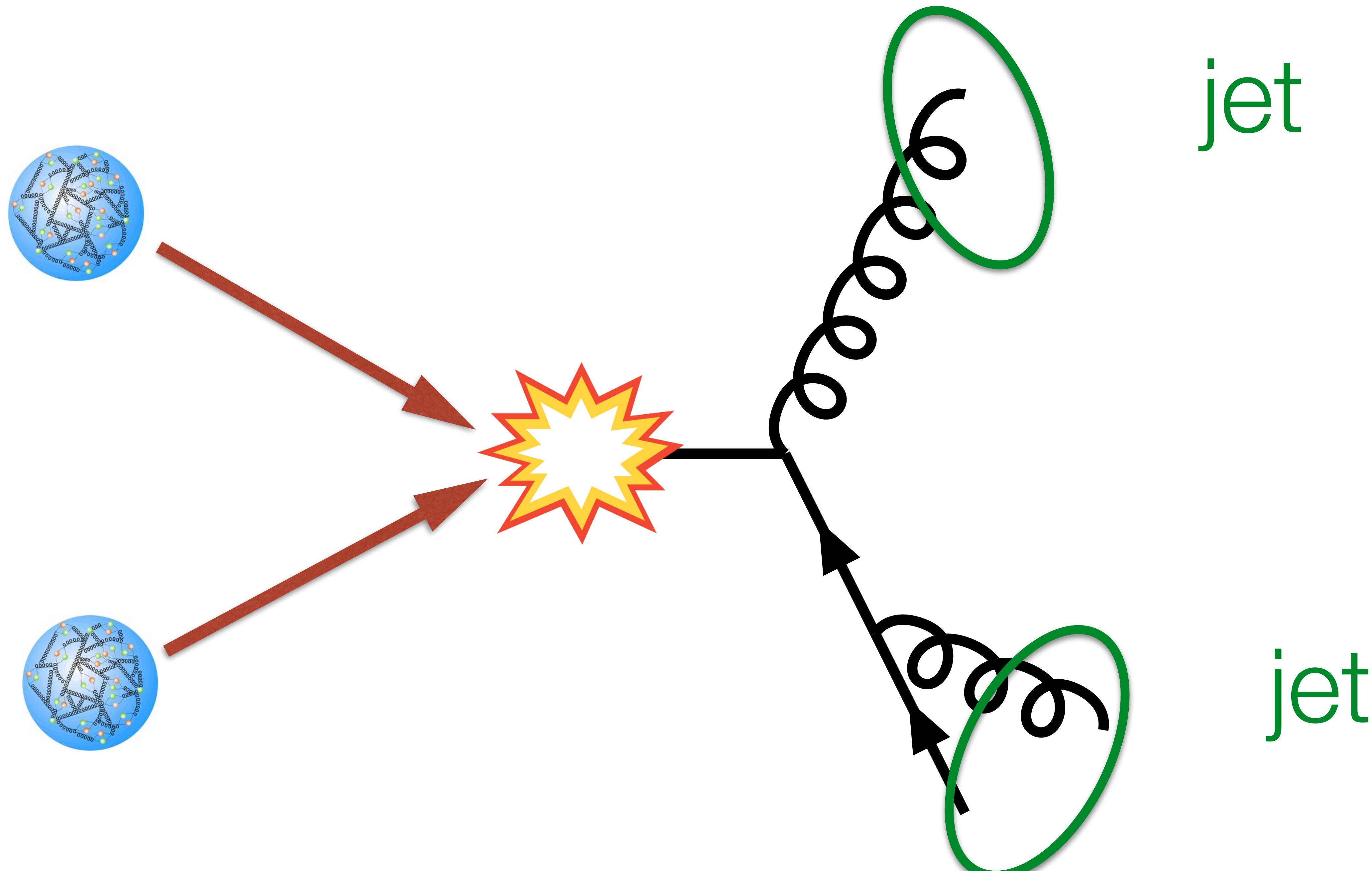
PDF factorization

We now discuss the factorization theorem at the heart of hadron collider physics. The factorization of the cross section into

- Parton Distribution Functions (PDFs) which describe the non-perturbative physics inside hadrons and
- perturbative hard scattering cross sections which describe the scattering of partons (quarks and gluons) at high energies

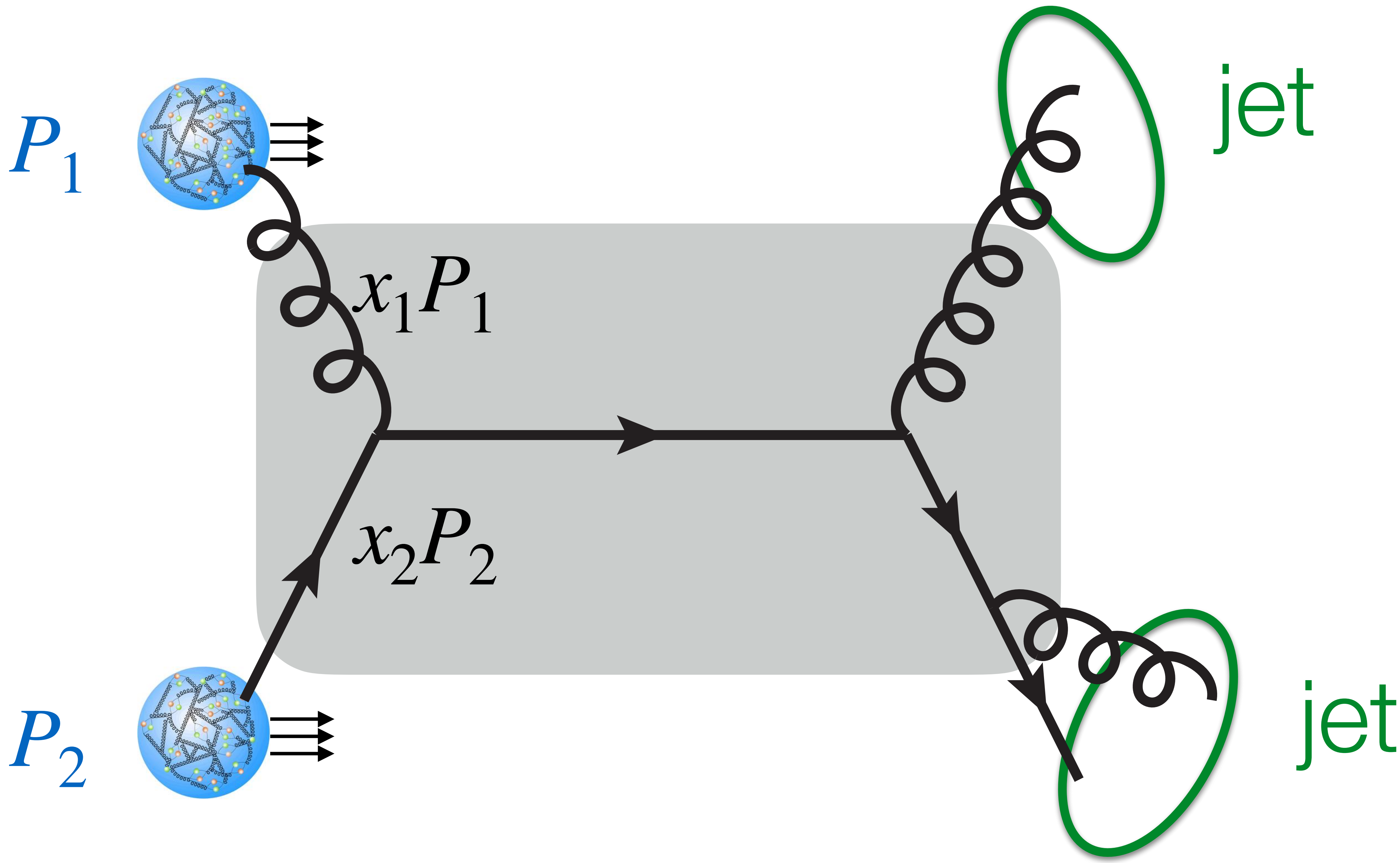


For e^+e^- collisions, we have demonstrated that for IR safe observables we can compute with quarks and gluons instead of hadrons in the final state.



Remains true at the hadron collider, but we also have non-pertubative dynamics in the initial-state hadrons!

Thanks to asymptotic freedom, high-energy scattering takes place between individual partons (quarks and gluons) inside the hadrons.



Non-perturbative Parton Distribution Functions (PDFs) give the probability to find parton (e.g. gluon, or an s-quark) with fraction x_i of the hadron momentum P_i . Hard scattering cross section of partons is evaluated in perturbation theory.

Factorization theorem

$$\sigma = \sum_{i,j \in q,\bar{q},g} \int_0^1 dx_1 \int_0^1 dx_2\, f_{i/h_1}(x_1, \mu) f_{j/h_2}(x_2, \mu)\, \hat{\sigma}_{ij}(\mu)$$

- PDF $f_{i/h}(x, \mu)$ depends on parton i hadron type h, momentum fraction x and factorization scale μ
- $\hat{\sigma}_{ij}$ is the partonic cross section for scattering of i and j into selected final state (jets, etc.)
- Theorem holds up to terms suppressed by powers of Λ_{QCD}/Q

Historical side remark

Computing hadronic cross sections with quarks and gluons inside proton is sometimes referred to as the parton model.

Historically, before QCD, Feynman and Bjorken observed that protons scattered like bags of point-like constituents ``partons".

Behavior is explained by asymptotic freedom of QCD and partons were later identified with quarks and gluons. This is sometimes called the QCD improved parton model, a horrific name, please don't use it!

Factorization is a theorem about scattering in QCD in high-E limit, not a model. Of course, theorems need to be proven…

for inclusive DY: Collins, Soper, Sterman '85

Parton distribution functions

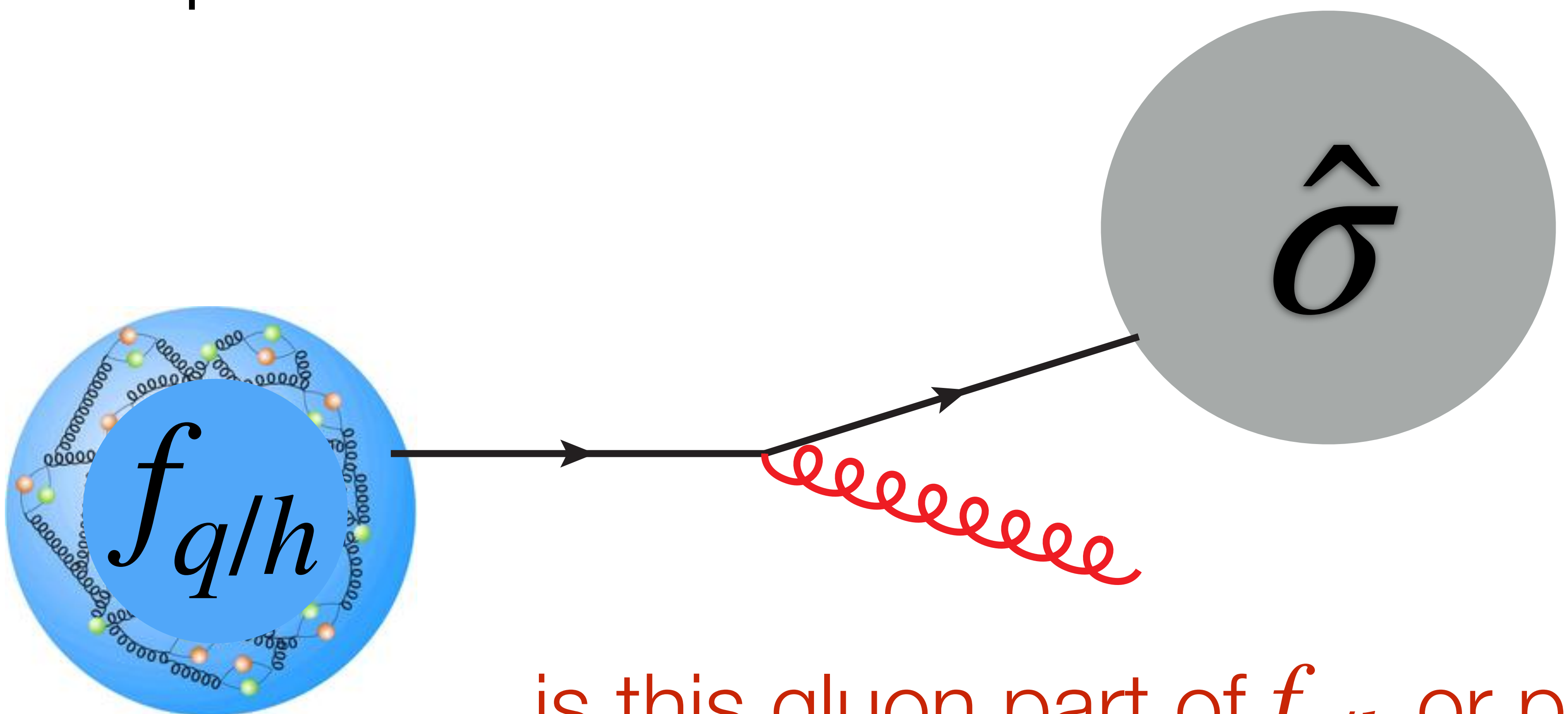
hadron state quark fields

$$f_{q/h}(x,\mu) = \frac{1}{4\pi}\int dt\, e^{-i\,x\,t\,n\cdot P}\,\langle h(P)\,|\,\bar{\chi}(tn)\,n_\mu\gamma^\mu\,\chi(0)\,|\,h(P)\rangle$$

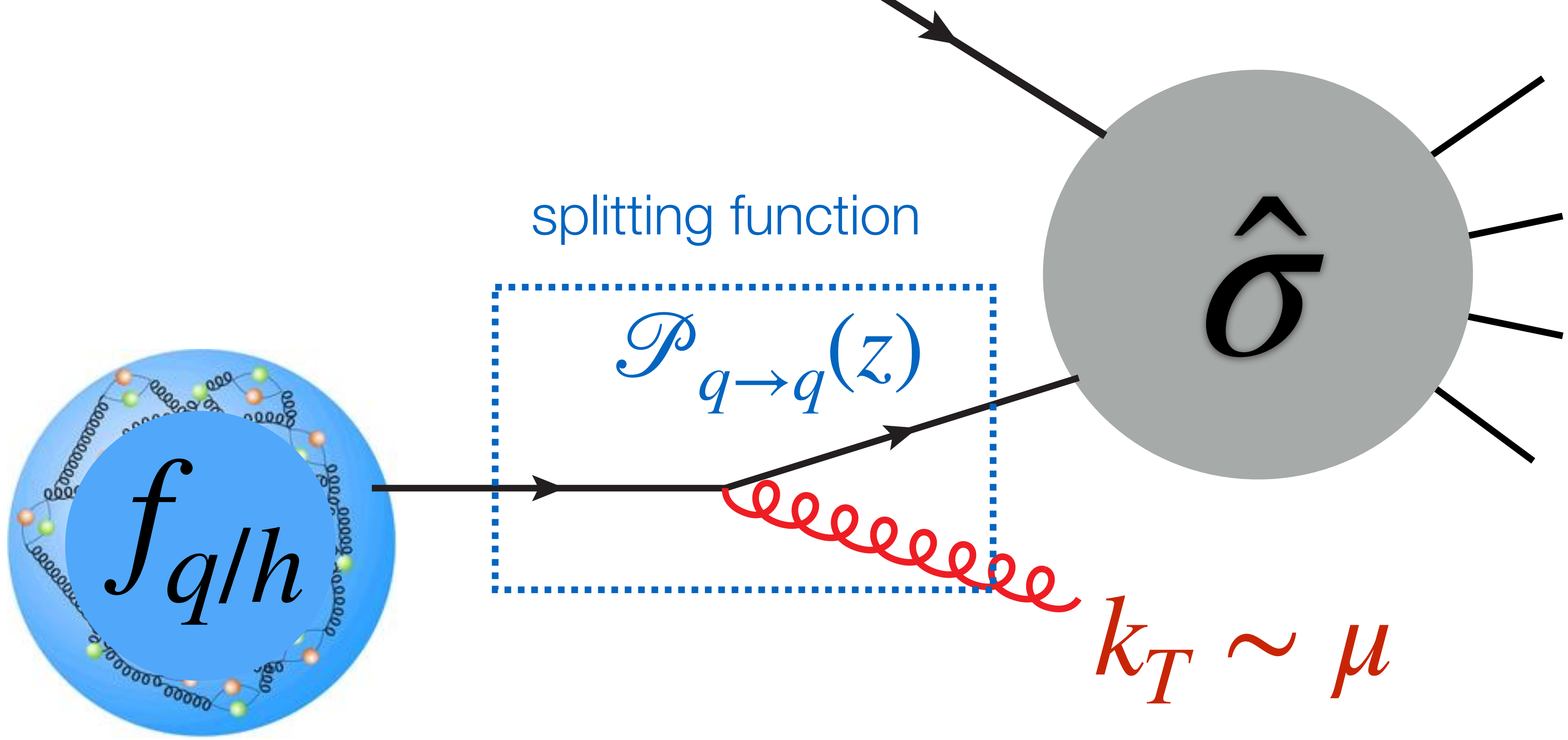
- Non-perturbative hadron matrix element of (quark or gluon) fields separated by light-like distance
  - n^μ is light-like reference vector, $n^2 = 0$
- progress in computing them using lattice QCD, but currently
- PDFs are determined from experimental measurements
  - parameterize functions $f_{i/h}(x,\mu)$ at reference scale $\mu = \mu_0$
  - perform global fit to measured cross sections

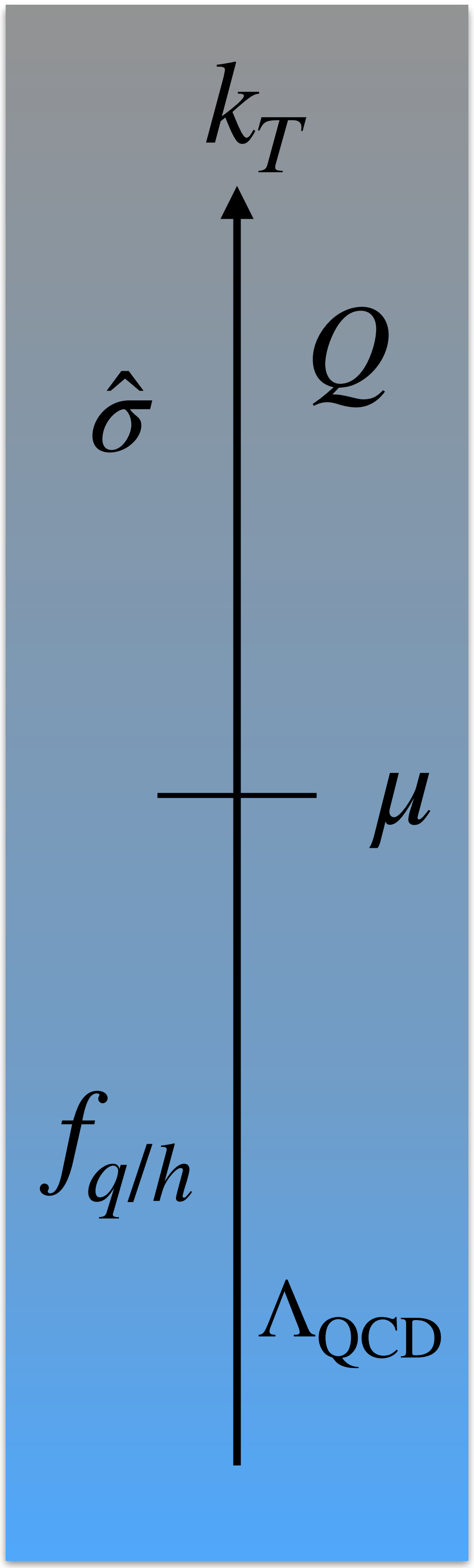
Factorization scale

PDFs and partonic cross section depend on a factorization scale μ . Defines separation into hard process $\hat{\sigma}$ and PDF



is this gluon part of $f_{q/h}$ or part of $\hat{\sigma}$??





- PDF matrix element gets UV divergences from energetic gluons with large $k_T \gg \Lambda_{\rm QCD}$
- partonic $\hat{\sigma}$ gets collinear divergences from partons with very low $k_T \ll Q$
- Subtraction at scale μ resolves both issues

DGLAP* evolution

Change in scale μ can be computed perturbatively

$$\frac{\partial f_{i/h}(x,\mu)}{\partial \ln \mu} = \sum_{j \in q,\bar{q},g} \frac{\alpha_s(\mu)}{\pi} \int_x^1 \frac{dz}{z} f_{j/h}\left(x/z,\mu\right) \mathscr{P}_{j\to i}(z)$$

- Evolution driven by (space-like) splitting functions $\mathscr{P}_{i\leftarrow j}(z)$. Higher order functions up to α_s^4 are known.
- Evolution is solved numerically.
- For $\hat{\sigma}(Q,\mu)$ must choose $\mu \sim Q$ to avoid logarithmic enhancements of higher orders.

* Dokshitzer–Gribov–Lipatov–Altarelli–Parisi

PDF determination

Fits from several groups (ABMP, CT, HERA,MSHT, NNPDF). differ by

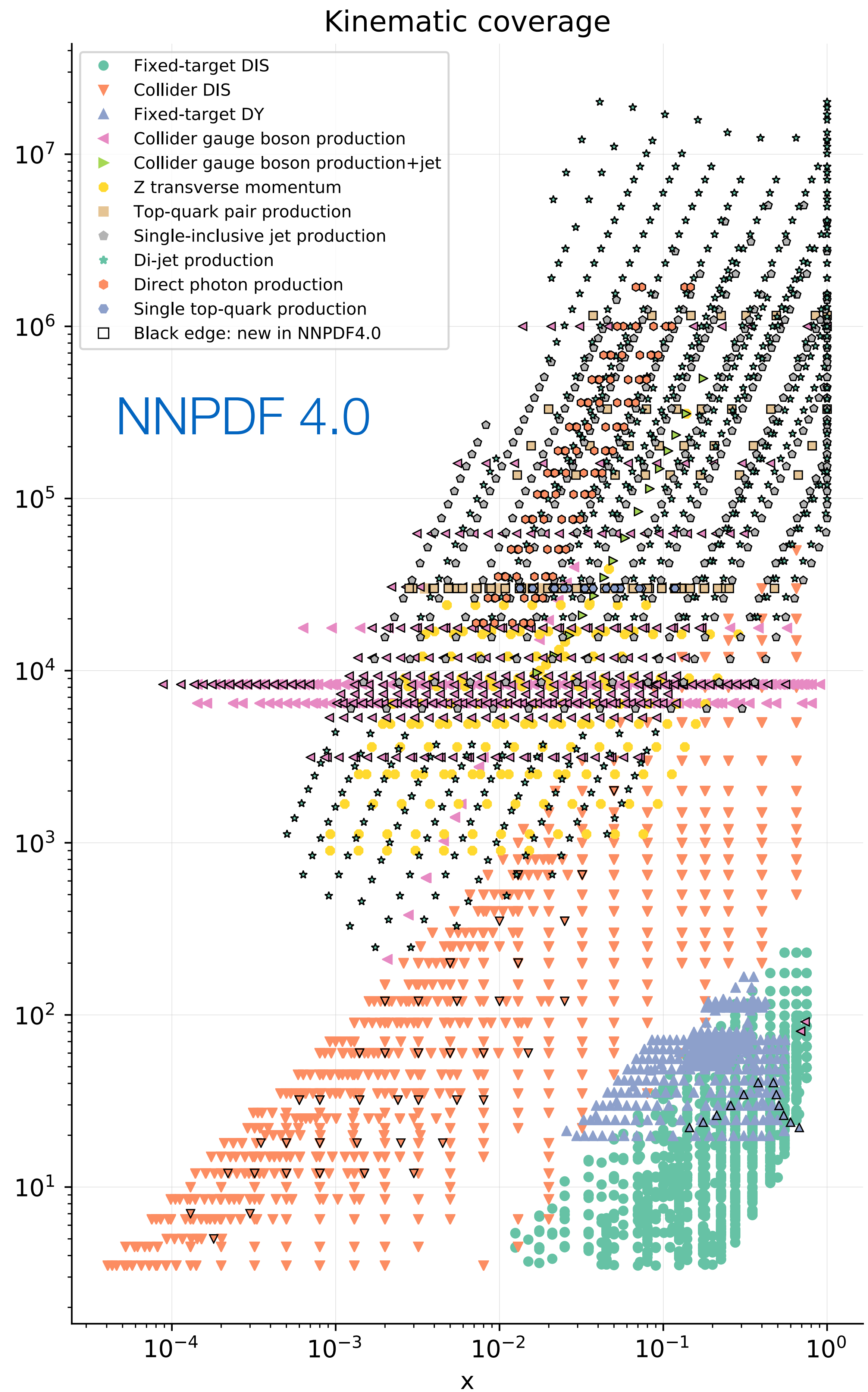
- Parameterization (fixed form, neural network)
- Method to estimate uncertainty (error eigenvector PDFs vs. replikas), theory uncertainties
- Data selection
- Treatment of heavy quark masses, α_s (fitted or fixed), QED effects, EW effects.

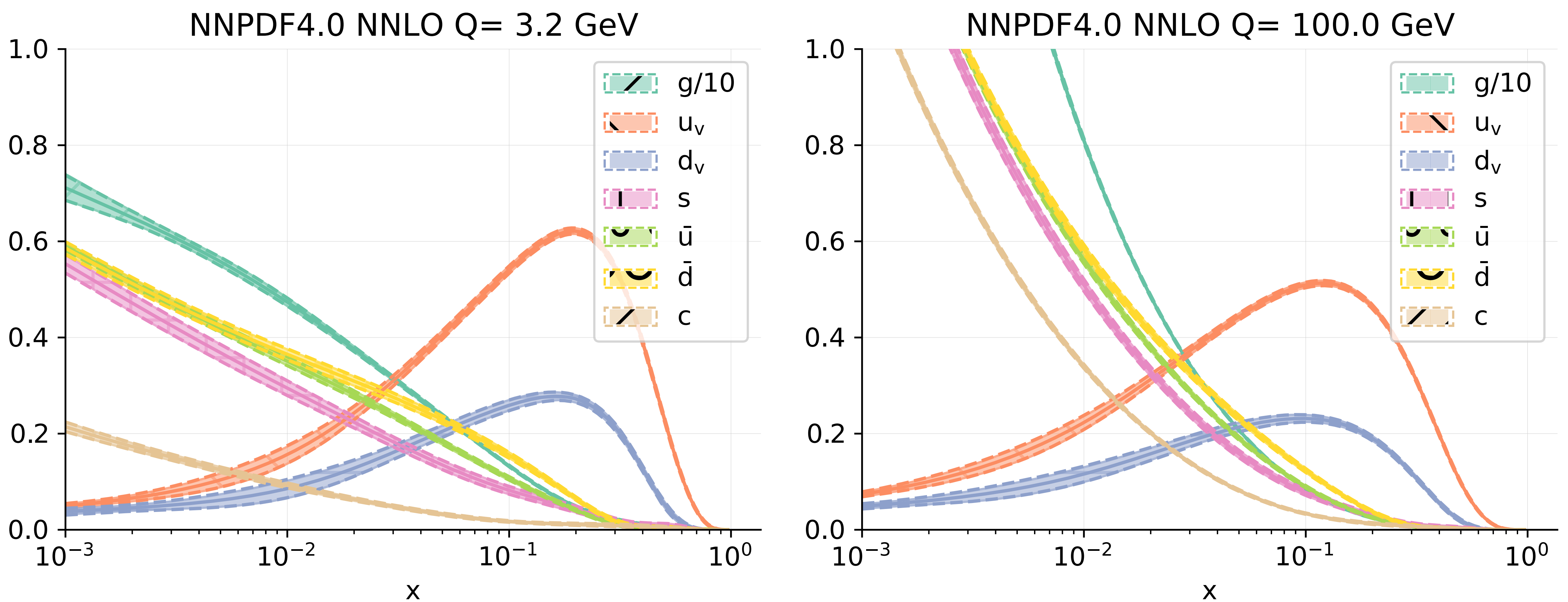
PDF sets for different orders and purposes (LO, NLO, NNLO, …, MC, resummation, BSM)

- Deep Inelastic Scattering (DIS) $e^- p \to e^- + X$ measurements at HERA are crucial

- also fixed-order DIS and DY at large x

- Fits need theoretical predictions for all processes at appropriate order.





- At large x, mostly u and d quarks. Proton ~ uud. PDFs vanish for $x \to 1$.
- At small x, gluons dominate. Also many valence quarks from $g \to q\bar{q}$. Most LHC collisions have small x.
- Uncertainty bands from replika sets. (NNPDF provides 1000 replikas.)

LHAPDF 6.5.5

Main page | PDF sets | Class hierarchy | Functions | Examples | More... | Search

LHAPDF Documentation

Introduction

LHAPDF is the standard tool for evaluating parton distribution functions (PDFs) in high-energy physics. PDFs encode the flavour and momentum structure of composite particles, such as protons, pions and nuclei; most cross-section calculations are based on parton-level matrix-elements which must be connected to the real interacting particles, hence PDFs are an essential ingredient of phenomenological and experimental studies at hadron and heavy-ion colliders (e.g. LHC, HERA, Tevatron, EIC, FCC) and in cosmic-ray physics.

PDFs themselves are fitted to a range of data by various collaborations. **LHAPDF** provides the definitive community library of such fits, in a standard data-format, as well as C++ and Python interfaces for evaluating them. Written as a general purpose C++ interpolator for estimating PDFs from discretised data files, it has also found more general uses, such as for fragmentation functions (essentially the inverse of PDFs).

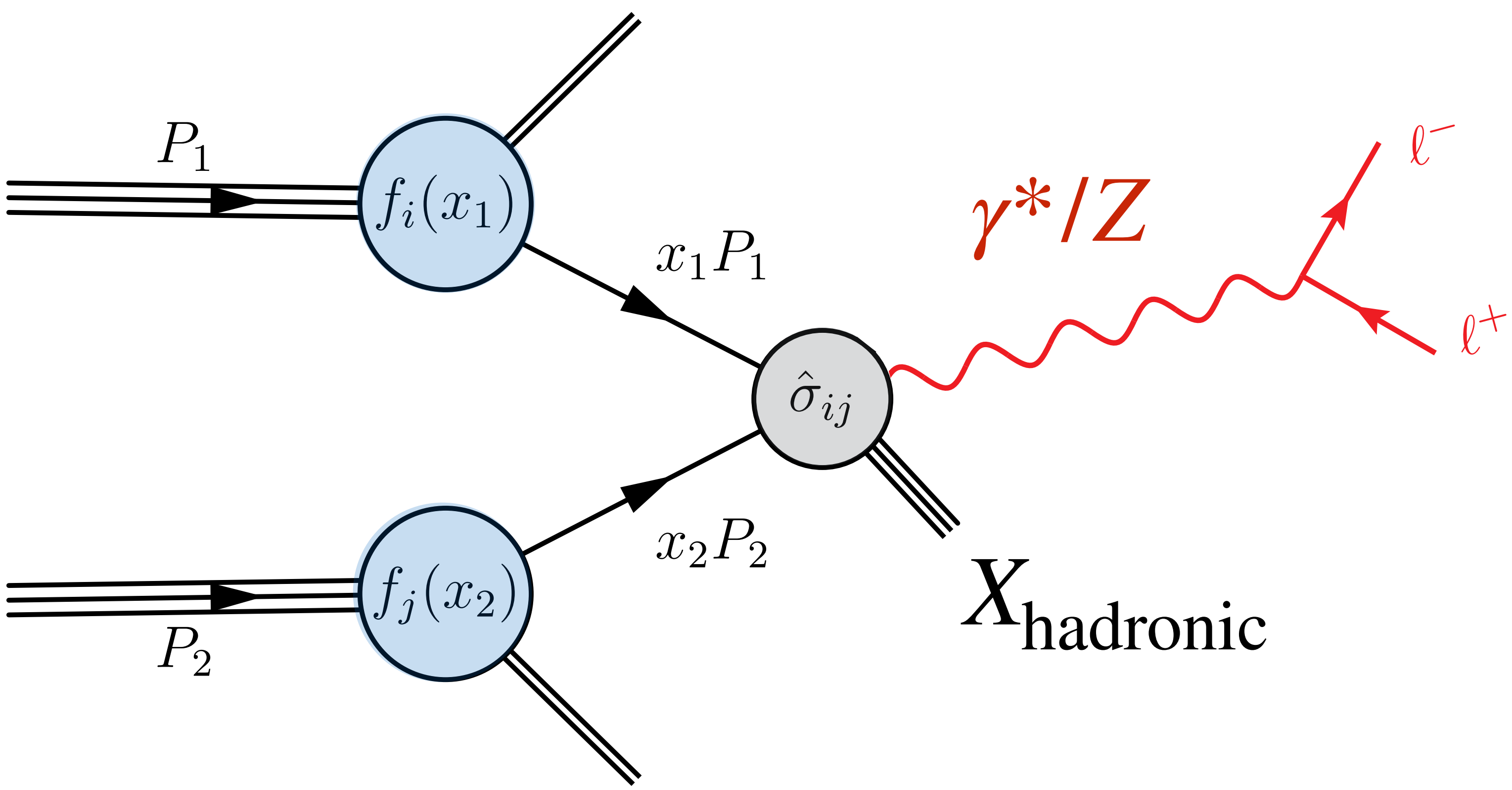
Table of Contents

- Introduction
- History and compatibility
- Installation
- Official PDF sets
  - Submitting new PDF sets
- Usage
  - Building against LHAPDF
  - Runtime symbol resolution
  - Trick to remove unwanted PDF members
  - Trick to use zipped data files
- Authors
- Support and bug reporting
- For developers

LHAPDF provides a common interface to PDF sets, used by many MC codes.

Drell-Yan process

The simplest hard-scattering process at a hadron collider is production of electroweak bosons

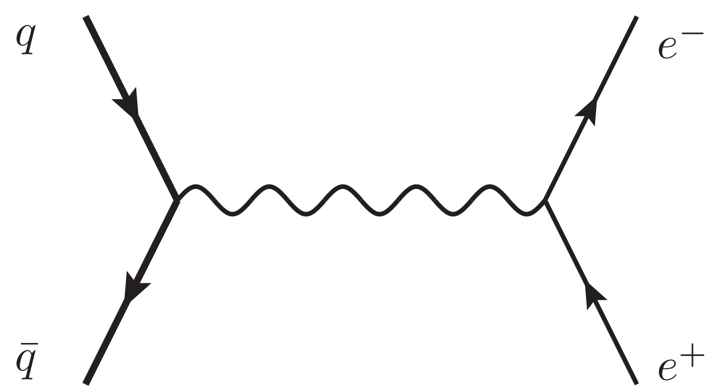


$q = p_{\ell^+} + p_{\ell^-}$

Boson kinematics:

$$Q^2 = q^2 \qquad \tau = \frac{Q^2}{s} \qquad y = \frac{1}{2} \ln \frac{y^0 + y^3}{y^0 - y^3} \qquad q_T^2 = q_x^2 + q_y^2$$

LO prediction



For γ-exchange

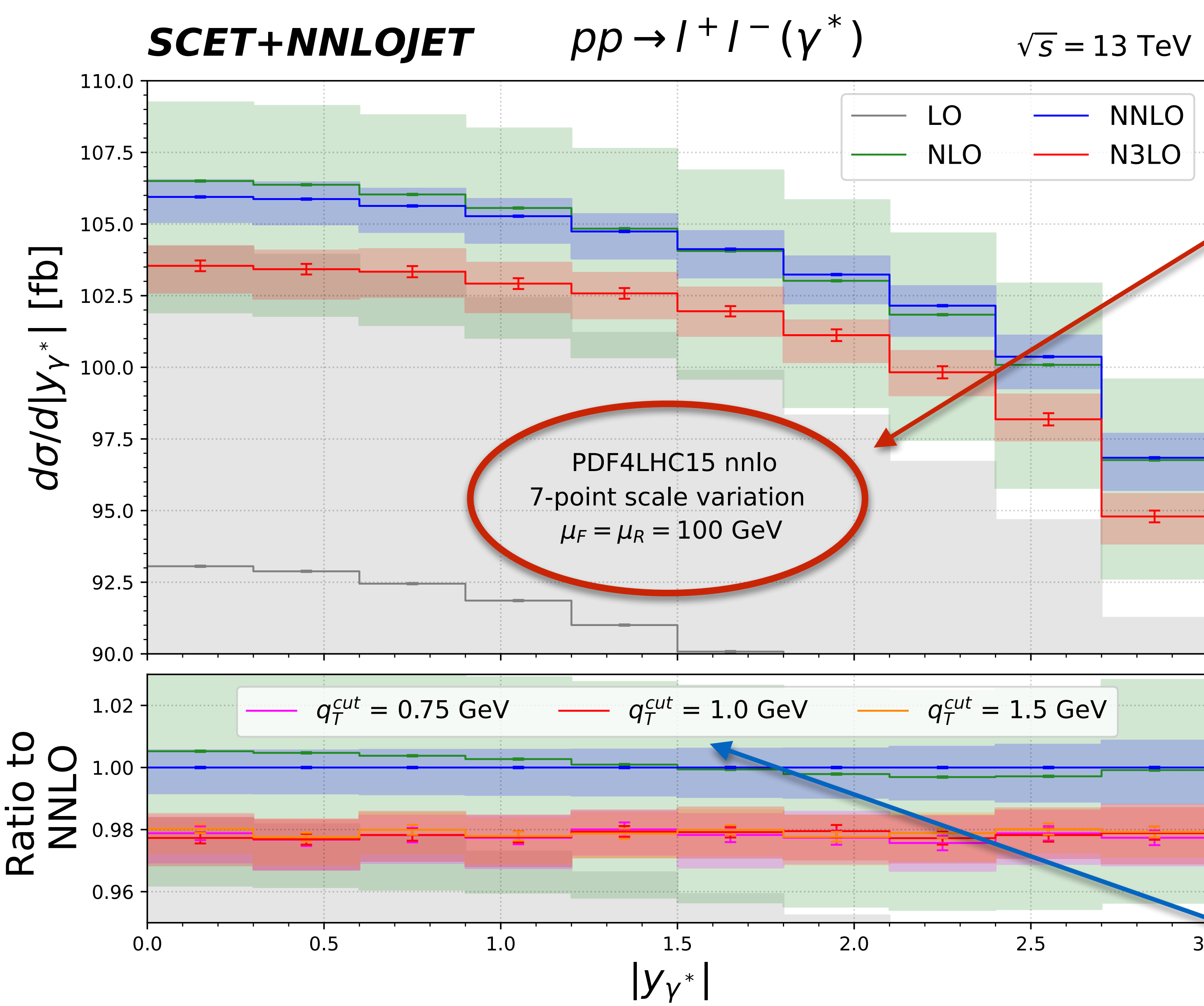
$$\hat{\sigma}_{q\bar{q}} = \frac{4\pi}{3}\frac{\alpha^2}{Q^2}\frac{Q_q^2}{N_c}$$

$$\frac{d^2\sigma}{dQ^2dy} = \sum_q \hat{\sigma}_{q\bar{q}} \left[f_q(x_1,\mu) f_{\bar{q}}(x_2,\mu) + (q \leftrightarrow \bar{q}) \right]$$

with $x_1 = \tau\, e^{y}$ and $x_2 = \tau\, e^{-y}$

- No transverse momentum at LO, $q_T = 0$.
- Rapidity distribution gives direct access to product of PDFs. (Interesting asymmetries in $W^{\pm}$ production!)

N^3LO predictions



Separate variation of

- scale μ_f in PDF
- scale μ_r in α_s

by factor two, subject to

$$1/2 < \mu_f/\mu_r < 2$$

→ 7 scale settings

Only NNLO PDFs available

$q_T^{\rm cut}$ is technical parameter in slicing method: lecture 4

2301.11827

Part IV

Monte Carlo Methods and Parton Showers

- Monte Carlo techniques
- Fixed-order MCs
- Parton showers

Monte-Carlo integration

Basic principle is to evaluate integrals by random sampling

$$I = \int_0^1 dx\, f(x) \to I_N = \frac{1}{N} \sum_{i=1}^{N} f(z_i)$$

where $z_i \in [0,1]$ are random numbers with flat distribution.
Uncertainty estimate from variance

$$I = I_N \pm \frac{\sigma}{\sqrt{N}} \quad \text{with} \quad \sigma^2 \approx \sigma_N^2 = \frac{1}{N} \sum_{i=1}^{N} f(z_i)^2 - I_N^2$$

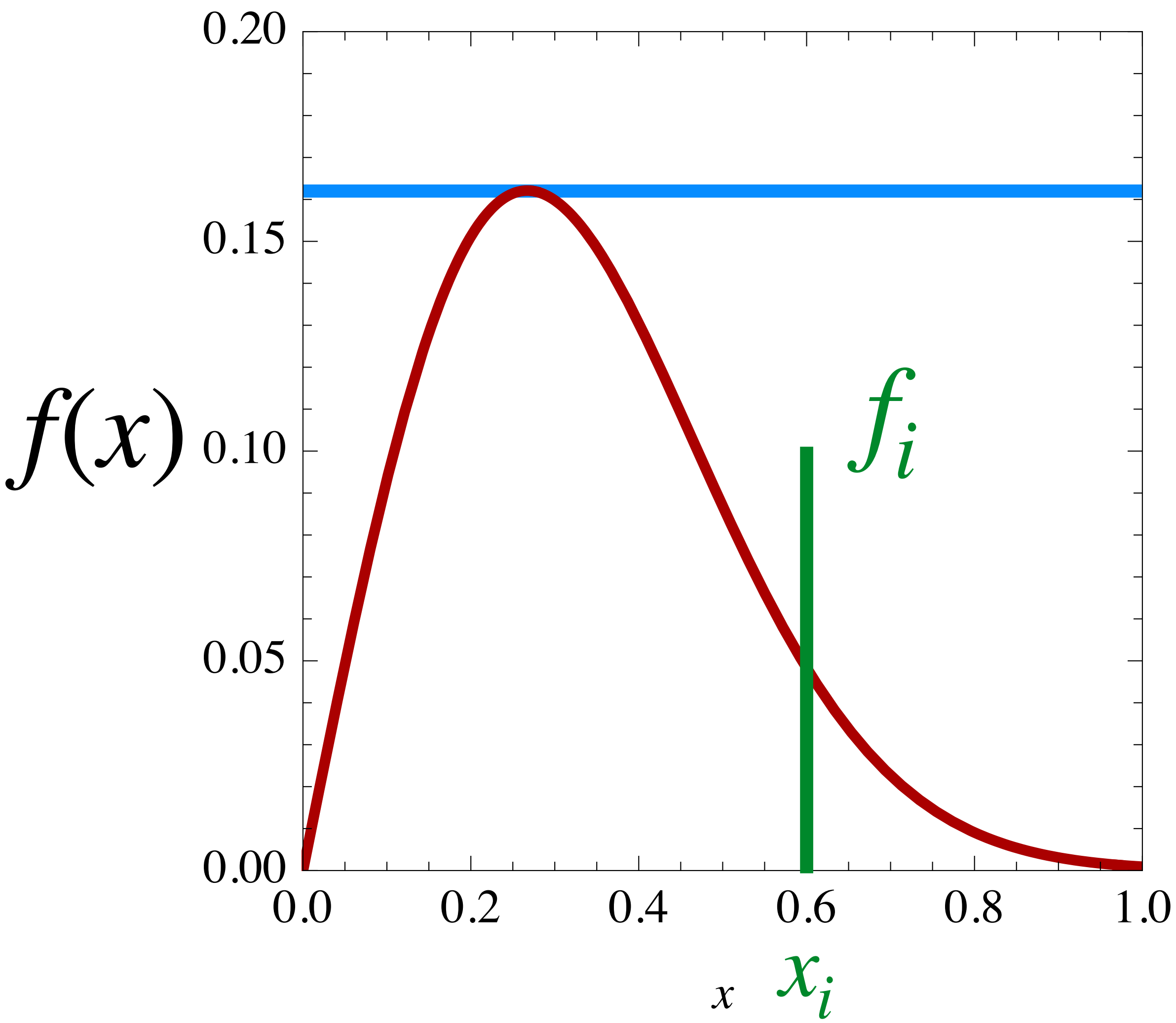
- Scales as $1/\sqrt{N}$; independent of dimension of integral
- Minimize variance (variable change) for accurate results (Exercise: try MC integration for $f(x) = 1/\sqrt{x}$)

Event generator

- MC method is used for phase-space integration. Dimension of integrals is $3n$-4 for n particles
- MC sample point $f(z_i)$ can be viewed a (collider) event with "weight" $w_i = f(z_i)$
- In nature, events have $w_i = 1$. Instead of a large function value $f(z_i)$, we get more events when the cross section is large, fewer when it is small
- Convenient to have a $w_i = 1$ event sample, then event generator behaves like a virtual collider

Unweighting

For a bounded function $0 \leq f(x) \leq f_{\max}$ we can obtain $w_i = 1$ events as follows



1. choose random $x_i \in [0, 1]$
2. choose random $f_i \in [0, f_{\max}]$
3. if $f_i < f(x_i)$ accept event

Fixed-order MC codes

- With appropriate cuts, differential tree-level cross sections are positive and bounded
  - Tree level $w = 1$ event generators
  - $w = 1$ parton showers
- Higher-order partonic cross sections are unbounded and do not have definite sign (negative virtual corrections)
  - Higher-order fixed-order MC codes can not provide $w = 1$ events
  - Only suitable integrals are IR finite and meaningful

Real emission IR singularities

Result for the $q\bar{q}g$ real emission phase space in $d = 4 - 2\varepsilon$ has the form

$$\sigma_{q\bar{q}g} = \int_0^1 dy_1 \int_0^{y_1} dy_2 \, y_1^{-1-\varepsilon} y_2^{-1-\varepsilon} f(y_1, y_2, \varepsilon) \, \mathcal{O}(y_1, y_2)$$

where $f(y_1, y_2)$ is regular. For IR safe $\mathcal{O}(y_1, y_2)$, poles in ε in $\sigma_{q\bar{q}g}$ cancel against those in loop corrections to $\sigma_{q\bar{q}}$.

Result in $d = 4 - 2\varepsilon$ is unsuitable for MC integration!

Solution:

- Extract IR singularities, combine with virtual.
- MC integral for finite reminder.

Toy integral

Virtual and real correction

$$V = \left(-\frac{1}{\varepsilon} + 2\right) f(0) \quad \text{and} \quad R = \int_0^1 dx \, \frac{1}{x^{1-\varepsilon}} \, f(x)$$

IR safety: $\lim_{x \to 0} f(x) = f(0)$

Methods to isolate divergences:

- subtraction
- slicing

Subtraction method

Subtract singular limit in integrand in

$$\overline{R} = R - S = \int_0^1 dx \, \frac{1}{x^{1-\varepsilon}} \, \left[f(x) - f(0) \right]$$

Subtraction term is evaluated analytically

$$S = f(0) \int_0^1 \frac{1}{x^{1-\varepsilon}} = f(0) \frac{1}{\varepsilon}$$

and added to virtual correction

$$\sigma_f^{\mathrm{NLO}} = R + V = (R - S) + (V + S) = \overline{R} + \overline{V}$$

- Both $\overline{R}$ and $\overline{V}$ are finite for $\epsilon = 0$.
- Real emission $\overline{R}$ can be evaluated with MC integration.

Slicing method

Isolate singular piece by splitting integration

$$R_\delta = \int_\delta^1 dx \, \frac{1}{x^{1-\varepsilon}} \, f(x)$$

Integrand is regular, bounded, positive, can set $\varepsilon = 0$ and use MC. In remainder, approximate $f(x) = f(0) + \mathcal{O}(x)$

$$S_\delta = f(0) \int_0^\delta \frac{1}{x^{1-\varepsilon}} + \mathcal{O}(\delta) = f(0) \left(\frac{1}{\varepsilon} + \ln \delta \right) + \mathcal{O}(\delta)$$

and add to virtual correction

$$\sigma_f^{\text{NLO}} = R_\delta + (S_\delta + V) + \mathcal{O}(\delta) = \sigma(x > \delta) + \sigma(x < \delta)$$

The two parts are physical cross sections!

Slicing

In contrast to subtraction, slicing involves an expansion in slicing parameter δ .

- Important that is δ small enough that power corrections in δ are negligible!
- Small δ is numerically difficult. Large cancellations between R_δ and $\sigma(x < \delta)$

Advantage of slicing is that it is done on the level of cross sections, which can be computed independently

$$\sigma = \sigma(x > \delta) + \sigma(x < \delta)$$

For NLO: tree-level cross section; compute with tree-level generator

expanded in δ and observable independent; compute using factorization theorem at small δ

NLO subtraction methods

Realistic NLO computation for n-jet cross section involves

- n-parton phase space for virtual corrections
- $(n+1)$-parton real emissions corrections
- Singularities when 2 partons become collinear, or 1 parton becomes soft

There are general algorithms for the subtraction terms, based on universal soft and collinear factorization

- FKS subtraction Frixione, Kunszt, Signer ‘95
- Dipole subtraction Catani Seymour ’98

Both schemes have been automated and implemented into numerical codes.

One-loop amplitudes

Passarino and Veltman '79 showed that all loop integrals can be decomposed in a small set of known scalar integrals

$$\mathcal{M}^{\text{1-loop}} = \sum_{i_0<i_1<i_2<i_3} d_{i_0 i_1 i_2 i_3} \text{Box}_{i_0 i_1 i_2 i_3} + \sum_{i_0<i_1<i_2} c_{i_0 i_1 i_2} \text{Triangle}_{i_0 i_1 i_2} + \sum_{i_0<i_1} b_{i_0 i_1} \text{Bubble}_{i_0 i_1} + \sum_{i_0} a_{i_0} \text{Tadpole}_{i_0} + R + \mathcal{O}(\epsilon)$$

$$D_i = (l + p_i)^2 - m_i^2$$

$$\text{Tadpole}_{i_0} = \int d^d l \frac{1}{D_{i_0}}$$

$$\text{Bubble}_{i_0 i_1} = \int d^d l \frac{1}{D_{i_0} D_{i_1}}$$

$$\text{Triangle}_{i_0 i_1 i_2} = \int d^d l \frac{1}{D_{i_0} D_{i_1} D_{i_2}}$$

$$\text{Box}_{i_0 i_1 i_2 i_3} = \int d^d l \frac{1}{D_{i_0} D_{i_1} D_{i_2} D_{i_3}}$$

However, original integral reduction method leads to numerical instabilities and is unsuitable for complicated processes.

Integral reduction

This problem has been solved. We now have automated methods to reliably compute also fairly complicated loop amplitudes

- Unitarity and on-shell methods Ossola, Papadopoulos, Pittau ’07; Ellis, Giele, Kunszt ‘07 with fully numerical evaluation based on reduction at the integrand level
- Improvements on ``traditional” reduction technique Denner, Dittmaier ’06, ‘11

NLO: one-loop amplitudes

Main **one-loop amplitude providers**:

- **BlackHat(https://blackhat.hepforge.org/)**
- **Collier (https://collier.hepforge.org)**
- **GoSam (https://gosam.hepforge.org)**
- **Golem95 (https://golem.hepforge.org/)**
- **Helac-NLO/Helac-1Loop (https://helac-phegas.web.cern.ch)**
- **Ninja (https://ninja.hepforge.org/)**
- **Njet (https://www.physik.hu-berlin.de/de/pep/tools/njet)**
- **NLOX (http://www.hep.fsu.edu/~nlox/)**
- **OpenLoops (https://openloops.hepforge.org)**
- **Recola/Recola2 (https://recola.hepforge.org)**

✓ Fast, automated generation and numerical evaluation of one-loop amplitudes

✓ Easy interface with Sherpa, Herwig, POWHEG, and others

✓ QCD only, or full SM (QCD+EW)

from Giulia Zanderighi's talk at Planck2025

NLO: general purpose tools

1. MadGraph5_aMC@NLO (https://launchpad.net/mg5amcnlo)

- Full automation of NLO QCD and EW
- Process generation via FeynRules/UFO. Supports parton showers via aMC@NLO

2. Sherpa+OpenLoops (https://sherpa.hepforge.org, https://openloops.hepforge.org)

- SHERPA handles phase space, subtraction, matching, and showering
- OpenLoops provides fast NLO matrix elements. Efficient for multi-leg processes

3. Herwig+Matchbox (https://herwig.hepforge.org)

- Herwig's Matchbox module enables automated NLO QCD corrections and matching
- Works with external amplitude providers (OpenLoops, MadGraph, etc.)

4. POWHEG-BOX (http://powhegbox.mib.infn.it)

- NLO with matching to parton showers (POWHEG method)
- Semi-automated; requires user input for new processes

5. MCFM (https://mcfm.fnal.gov)

- Parton-level code for NLO calculations (less automated)
- Mostly SM processes. Mostly based on analytic calculations, very stable and fast

…

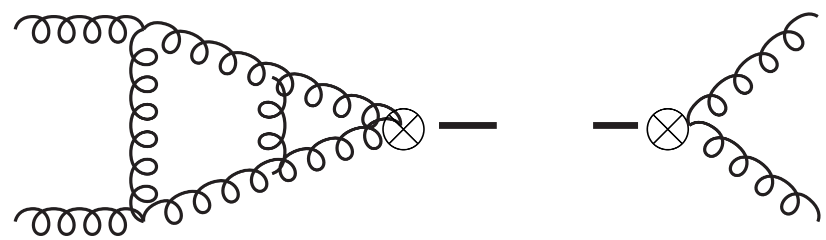
from Giulia Zanderighi's talk at Planck2025

Process		Syntax	Cross section (pb)					
Single Higgs production			LO 13 TeV			NLO 13 TeV		
g.1	$pp \to H$ (HEFT)	`p p > h`	$1.593 \pm 0.003 \cdot 10^{1}$	$^{+34.8\%}_{-26.0\%}$	$^{+1.2\%}_{-1.7\%}$	$3.261 \pm 0.010 \cdot 10^{1}$	$^{+20.2\%}_{-17.9\%}$	$^{+1.1\%}_{-1.6\%}$
g.2	$pp \to Hj$ (HEFT)	`p p > h j`	$8.367 \pm 0.003 \cdot 10^{0}$	$^{+39.4\%}_{-26.4\%}$	$^{+1.2\%}_{-1.4\%}$	$1.422 \pm 0.006 \cdot 10^{1}$	$^{+18.5\%}_{-16.6\%}$	$^{+1.1\%}_{-1.4\%}$
g.3	$pp \to Hjj$ (HEFT)	`p p > h j j`	$3.020 \pm 0.002 \cdot 10^{0}$	$^{+59.1\%}_{-34.7\%}$	$^{+1.4\%}_{-1.7\%}$	$5.124 \pm 0.020 \cdot 10^{0}$	$^{+20.7\%}_{-21.0\%}$	$^{+1.3\%}_{-1.5\%}$
g.4	$pp \to Hjj$ (VBF)	`p p > h j j $$ w+ w- z`	$1.987 \pm 0.002 \cdot 10^{0}$	$^{+1.7\%}_{-2.0\%}$	$^{+1.9\%}_{-1.4\%}$	$1.900 \pm 0.006 \cdot 10^{0}$	$^{+0.8\%}_{-0.9\%}$	$^{+2.0\%}_{-1.5\%}$
g.5	$pp \to Hjjj$ (VBF)	`p p > h j j j $$ w+ w- z`	$2.824 \pm 0.005 \cdot 10^{-1}$	$^{+15.7\%}_{-12.7\%}$	$^{+1.5\%}_{-1.0\%}$	$3.085 \pm 0.010 \cdot 10^{-1}$	$^{+2.0\%}_{-3.0\%}$	$^{+1.5\%}_{-1.1\%}$
g.6	$pp \to HW^{\pm}$	`p p > h wpm`	$1.195 \pm 0.002 \cdot 10^{0}$	$^{+3.5\%}_{-4.5\%}$	$^{+1.9\%}_{-1.5\%}$	$1.419 \pm 0.005 \cdot 10^{0}$	$^{+2.1\%}_{-2.6\%}$	$^{+1.9\%}_{-1.4\%}$
g.7	$pp \to HW^{\pm}j$	`p p > h wpm j`	$4.018 \pm 0.003 \cdot 10^{-1}$	$^{+10.7\%}_{-9.3\%}$	$^{+1.2\%}_{-0.9\%}$	$4.842 \pm 0.017 \cdot 10^{-1}$	$^{+3.6\%}_{-3.7\%}$	$^{+1.2\%}_{-1.0\%}$
g.8*	$pp \to HW^{\pm}jj$	`p p > h wpm j j`	$1.198 \pm 0.016 \cdot 10^{-1}$	$^{+26.1\%}_{-19.4\%}$	$^{+0.8\%}_{-0.6\%}$	$1.574 \pm 0.014 \cdot 10^{-1}$	$^{+5.0\%}_{-6.5\%}$	$^{+0.9\%}_{-0.6\%}$
g.9	$pp \to HZ$	`p p > h z`	$6.468 \pm 0.008 \cdot 10^{-1}$	$^{+3.5\%}_{-4.5\%}$	$^{+1.9\%}_{-1.4\%}$	$7.674 \pm 0.027 \cdot 10^{-1}$	$^{+2.0\%}_{-2.5\%}$	$^{+1.9\%}_{-1.4\%}$
g.10	$pp \to HZj$	`p p > h z j`	$2.225 \pm 0.001 \cdot 10^{-1}$	$^{+10.6\%}_{-9.2\%}$	$^{+1.1\%}_{-0.8\%}$	$2.667 \pm 0.010 \cdot 10^{-1}$	$^{+3.5\%}_{-3.6\%}$	$^{+1.1\%}_{-0.9\%}$
g.11*	$pp \to HZjj$	`p p > h z j j`	$7.262 \pm 0.012 \cdot 10^{-2}$	$^{+26.2\%}_{-19.4\%}$	$^{+0.7\%}_{-0.6\%}$	$8.753 \pm 0.037 \cdot 10^{-2}$	$^{+4.8\%}_{-6.3\%}$	$^{+0.7\%}_{-0.6\%}$
g.12*	$pp \to HW^{+}W^{-}$ (4f)	`p p > h w+ w-`	$8.325 \pm 0.139 \cdot 10^{-3}$	$^{+0.0\%}_{-0.3\%}$	$^{+2.0\%}_{-1.6\%}$	$1.065 \pm 0.003 \cdot 10^{-2}$	$^{+2.5\%}_{-1.9\%}$	$^{+2.0\%}_{-1.5\%}$
g.13*	$pp \to HW^{\pm}\gamma$	`p p > h wpm a`	$2.518 \pm 0.006 \cdot 10^{-3}$	$^{+0.7\%}_{-1.4\%}$	$^{+1.9\%}_{-1.5\%}$	$3.309 \pm 0.011 \cdot 10^{-3}$	$^{+2.7\%}_{-2.0\%}$	$^{+1.7\%}_{-1.4\%}$
g.14*	$pp \to HZW^{\pm}$	`p p > h z wpm`	$3.763 \pm 0.007 \cdot 10^{-3}$	$^{+1.1\%}_{-1.5\%}$	$^{+2.0\%}_{-1.6\%}$	$5.292 \pm 0.015 \cdot 10^{-3}$	$^{+3.9\%}_{-3.1\%}$	$^{+1.8\%}_{-1.4\%}$
g.15*	$pp \to HZZ$	`p p > h z z`	$2.093 \pm 0.003 \cdot 10^{-3}$	$^{+0.1\%}_{-0.6\%}$	$^{+1.9\%}_{-1.5\%}$	$2.538 \pm 0.007 \cdot 10^{-3}$	$^{+1.9\%}_{-1.4\%}$	$^{+2.0\%}_{-1.5\%}$
g.16	$pp \to Ht\bar{t}$	`p p > h t t~`	$3.579 \pm 0.003 \cdot 10^{-1}$	$^{+30.0\%}_{-21.5\%}$	$^{+1.7\%}_{-2.0\%}$	$4.608 \pm 0.016 \cdot 10^{-1}$	$^{+5.7\%}_{-9.0\%}$	$^{+2.0\%}_{-2.3\%}$
g.17	$pp \to Htj$	`p p > h tt j`	$4.994 \pm 0.005 \cdot 10^{-2}$	$^{+2.4\%}_{-4.2\%}$	$^{+1.2\%}_{-1.3\%}$	$6.328 \pm 0.022 \cdot 10^{-2}$	$^{+2.9\%}_{-1.8\%}$	$^{+1.5\%}_{-1.6\%}$
g.18	$pp \to Hb\bar{b}$ (4f)	`p p > h b b~`	$4.983 \pm 0.002 \cdot 10^{-1}$	$^{+28.1\%}_{-21.0\%}$	$^{+1.5\%}_{-1.8\%}$	$6.085 \pm 0.026 \cdot 10^{-1}$	$^{+7.3\%}_{-9.6\%}$	$^{+1.6\%}_{-2.0\%}$
g.19	$pp \to Ht\bar{t}j$	`p p > h t t~ j`	$2.674 \pm 0.041 \cdot 10^{-1}$	$^{+45.6\%}_{-29.2\%}$	$^{+2.6\%}_{-2.9\%}$	$3.244 \pm 0.025 \cdot 10^{-1}$	$^{+3.5\%}_{-8.7\%}$	$^{+2.5\%}_{-2.9\%}$
g.20*	$pp \to Hb\bar{b}j$ (4f)	`p p > h b b~ j`	$7.367 \pm 0.002 \cdot 10^{-2}$	$^{+45.6\%}_{-29.1\%}$	$^{+1.8\%}_{-2.1\%}$	$9.034 \pm 0.032 \cdot 10^{-2}$	$^{+7.9\%}_{-11.0\%}$	$^{+1.8\%}_{-2.2\%}$

sample results from MadGraph5_aMC@NLO Alwall et al. ’14 (paper now has >9700 citations)

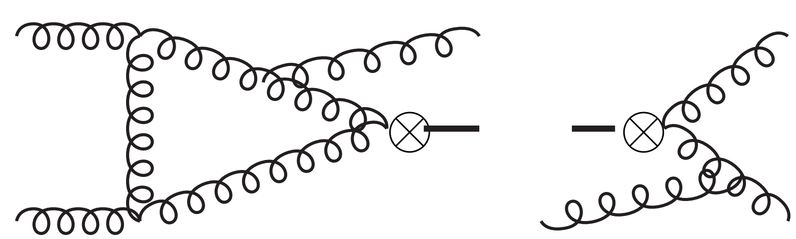
NNLO ingredients

- Two-loop virtual



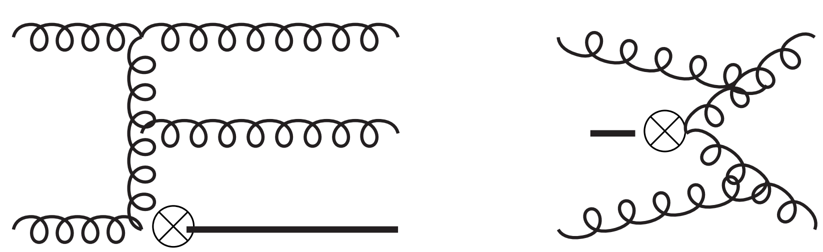
$+ 148$ terms;

- Real-virtual



$+ 635$ terms

- Double-real



$+ 594$ terms

(NNLO Higgs production Anastasiou and Melnikov ’02)

NNLO subtraction

Challenging structure of IR singularities at NNLO

- Double-soft, triple-collinear, soft-collinear, …
- Difficult to find have subtraction terms covering all limits that can be integrated analytically.

Many methods, all used in particular NNLO computations

- Antenna subtraction, sector improved residue subtraction, nested soft–collinear subtraction, local analytic subtraction, ColourFull subtraction, projection to Born, …

Still a very active area of research.

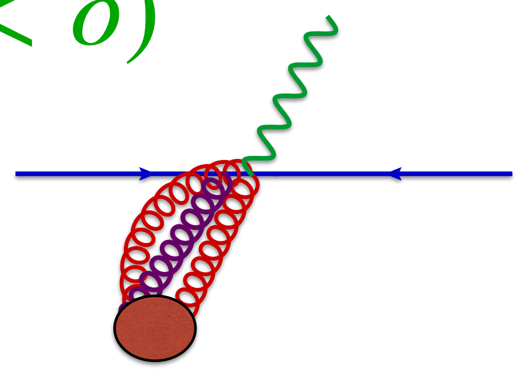
NNLO slicing

Catani, Grazzini ’07, Boughezal, Liu, Petreillo 15, Gaunt, Stahlhofen, Tackmann Walsh 15

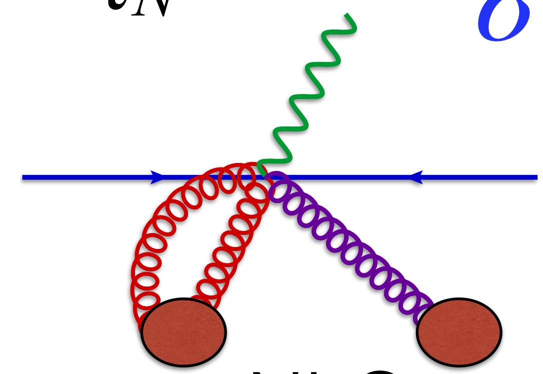
$\sigma(x < \delta)$

$x = q_{\rm T}$ or $x = \tau_N$

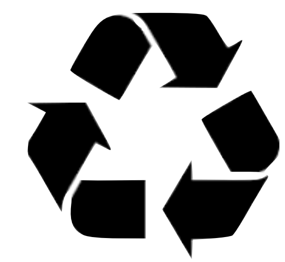
$\sigma(x > \delta)$



Two-loop graphs
Soft and collinear
emissions: factorization



NLO



Use transverse momentum q_T or event shape τ_N (N-jettiness) to separate out most singular region of NNLO computation

- Factorization theorems to compute $\sigma(x < \delta)$ in singular region
- Existing NLO codes away from end-point for $\sigma(x > \delta)$

Used widely, especially for electroweak boson production processes (also at N3LO!) and for boson + jet processes.

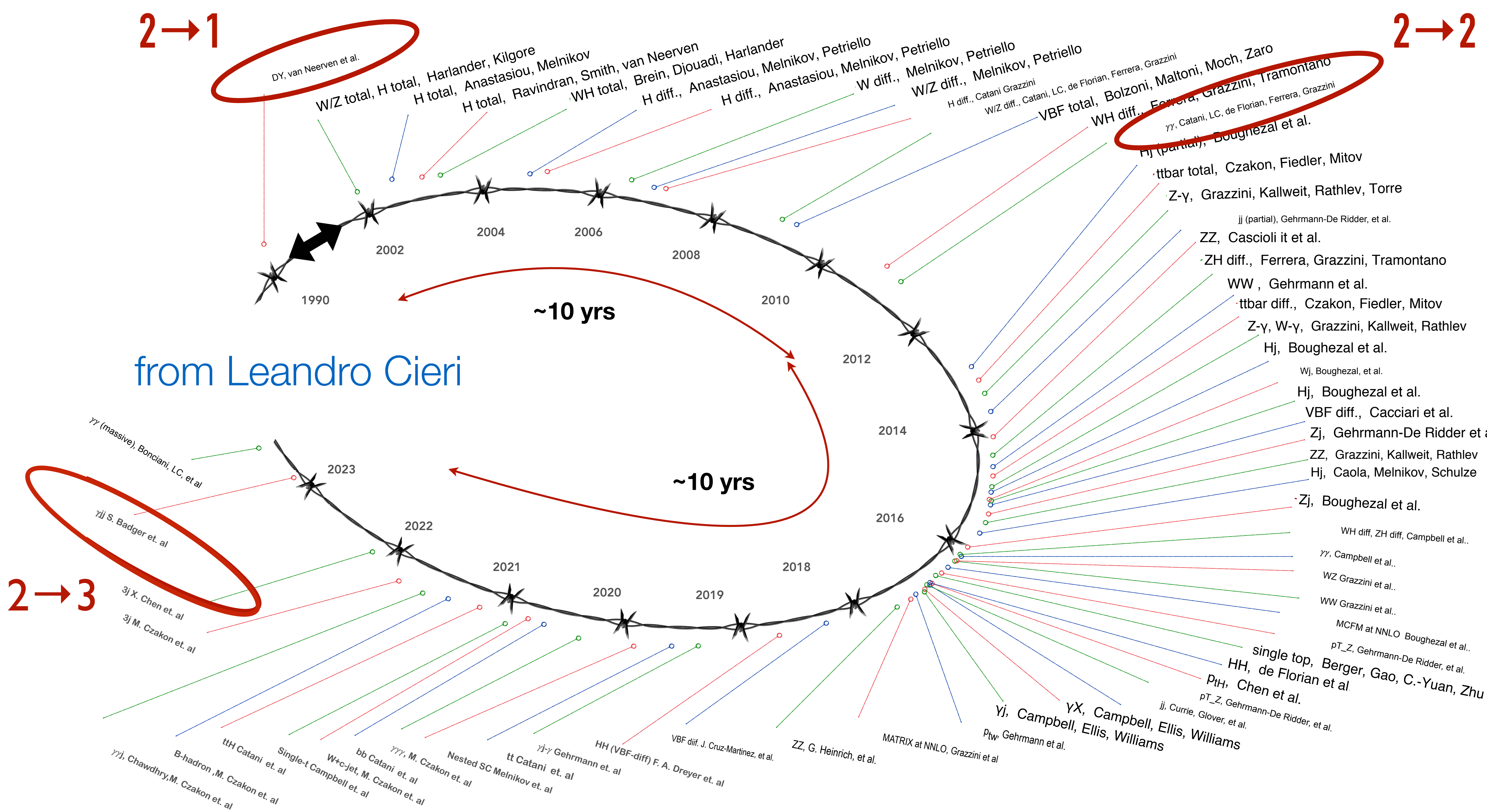
Two loop amplitudes

Big challenge at NNLO is the computation of two-loop integrals. Main strategy is

- reduction of loop integrals for a given process to a small set of master integrals using IBP identities
- analytic evaluation of the master integrals using differential equations, difference equations, Mellin–Barnes representations, Method of Regions expansions, iterated integrals…
- or numerical methods such as sector decomposition or auxiliary mass flow

Many new developments. Current frontier is $2 \to 2$ with masses/off-shell legs, massless $2 \to 3$.

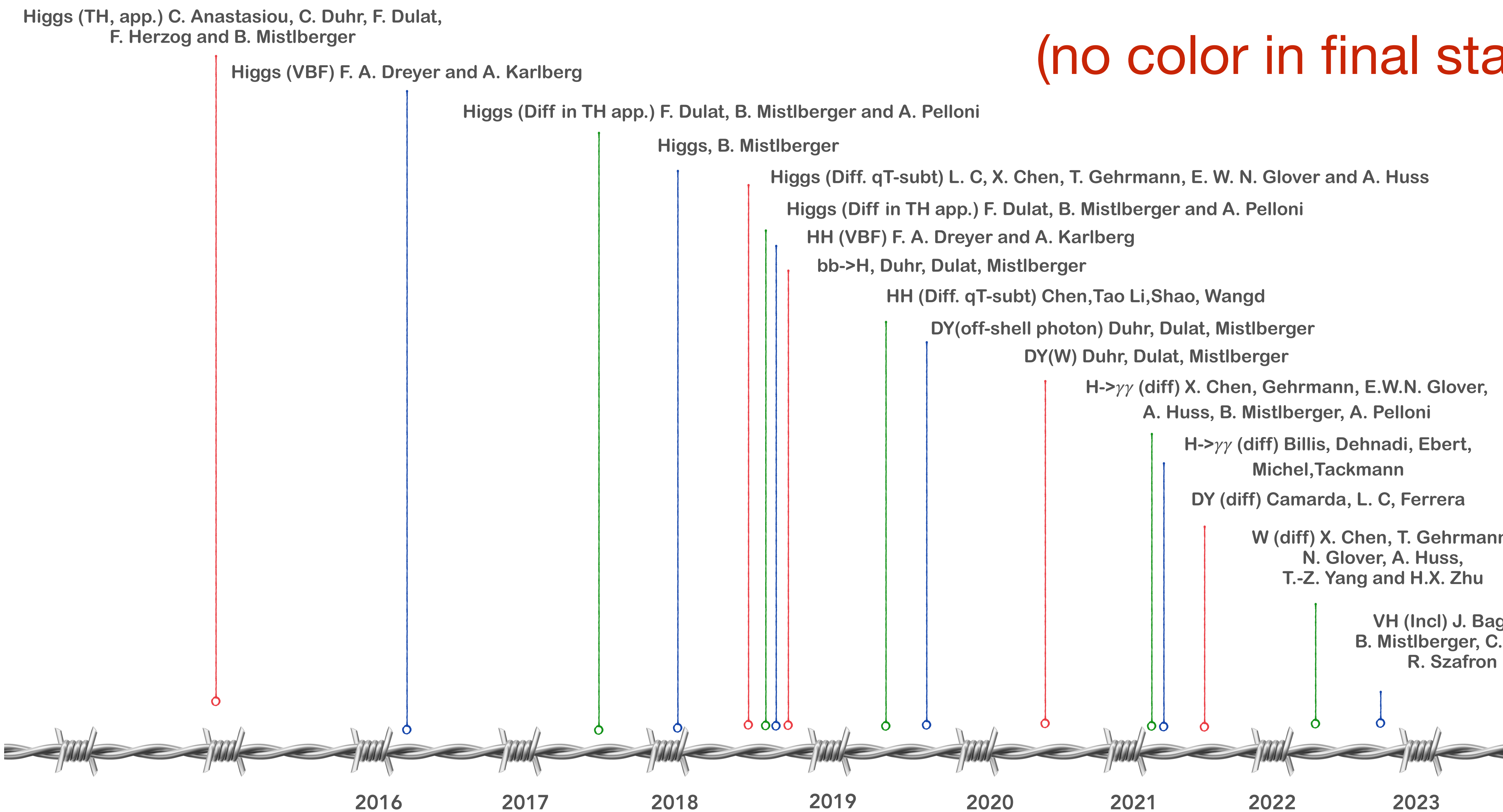
NNLO results



10 years / additional leg. No full automation yet!

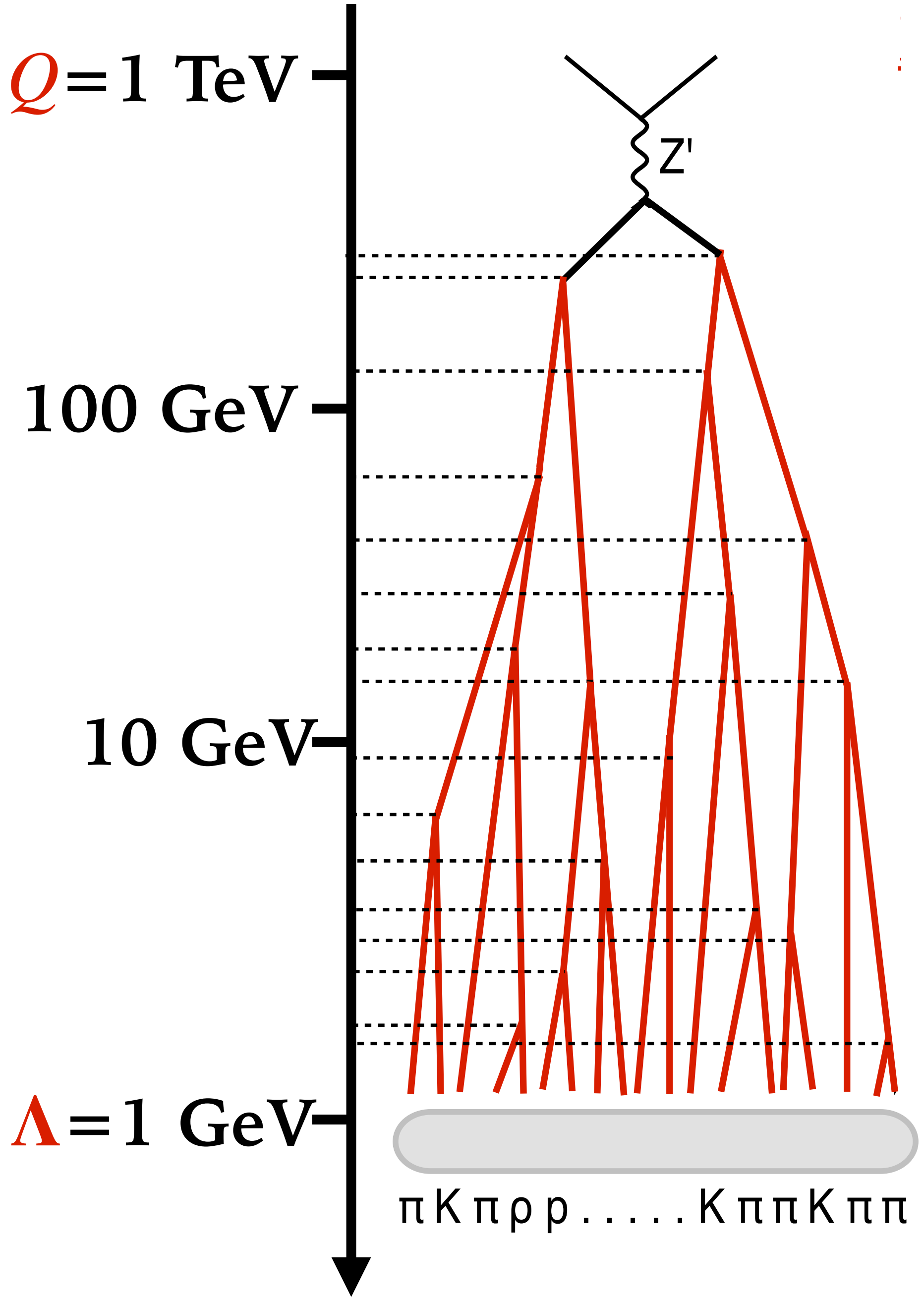
Status at N^{3}LO

Drell-Yan like 2→1
(no color in final state)



from Leandro Cieri

Parton shower MCs



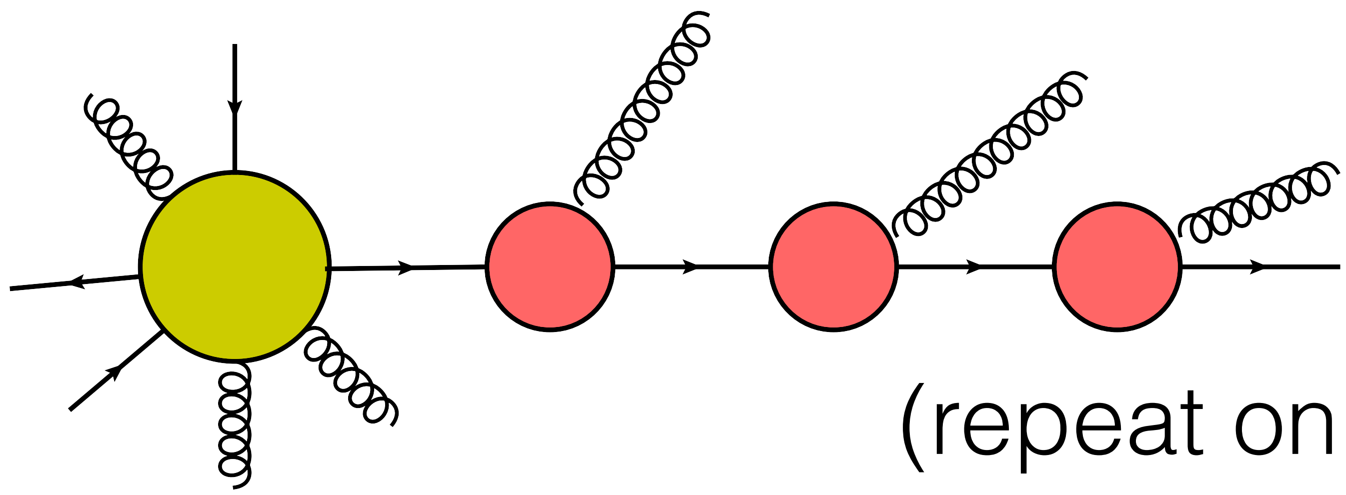
from Silvia Ferrario Ravasio

A crucial tool that can produce realistic collider events. Two main elements

- cascade of quark and gluon emissions down to low scale, approximate cross sections, based on collinear and soft factorization
- hadronisation model at the low scale
- + many additional ingredients. Hadron decays, MPI, QED, …

Two basic types

A. $1 \to 2$ branchings. Independent emissions from each leg



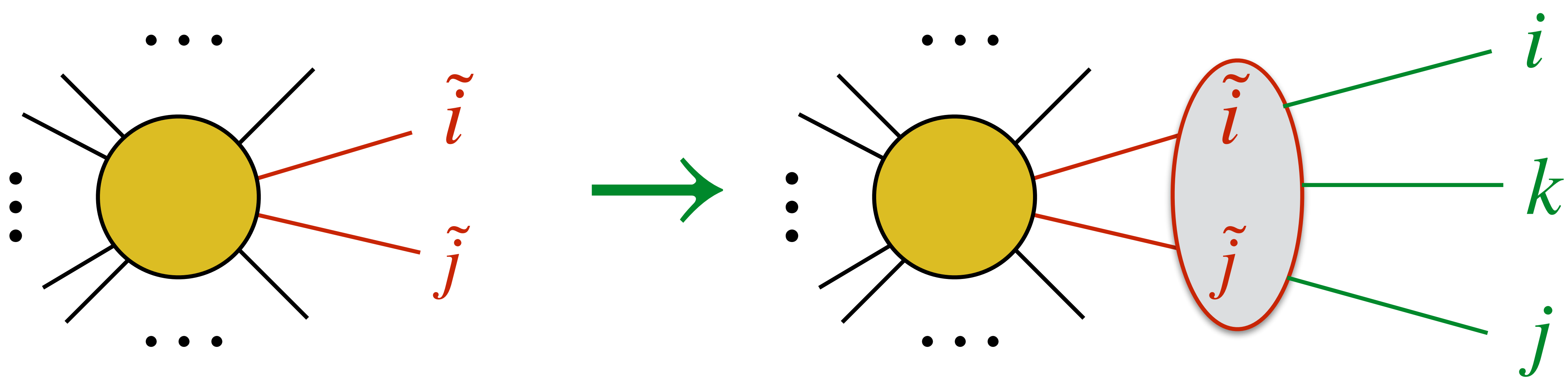
(repeat on each leg)

based on collinear factorization. Use angular ordering Marchesini, Webber '88 to get correct soft radiation pattern for simple observables.

Implemented in Herwig parton shower.

Two basic types

B. Dipole showers based on $2 \to 3$ branchings.



LO soft emission is sum of dipoles!

Dipoles/antennas capture both soft and collinear limit at LO and produce both types of enhancements: NLL accuracy is possible!

Well suited for matching to fixed order. Basis of most modern showers.

Recoil scheme

Soft and collinear factorization is based on expansions in these limits, e.g.

$$p_1 + \ldots + p_n + k_{\text{soft}} \approx p_1 + \ldots + p_n$$

Parton showers instead distribute recoil to have exact momentum conservation in each emission.

Two classes of prescriptions

- Local recoil: distribute recoil inside dipole. Modify $\tilde{p}_i \to p_i$ and $\tilde{p}_j \to p_j$ to ensure

$$\tilde{p}_i + \tilde{p}_j = p_i + p_j + k$$

- Global recoil: absorb k_T into all partons, also those not involved in the splitting.

Recoil and logarithmic accuracy

The recoil prescription can violate the scale separation underlying soft-collinear factorization.

For this reason parton showers such as Pythia, Herwig and Sherpa do not achieve full NLL accuracy.

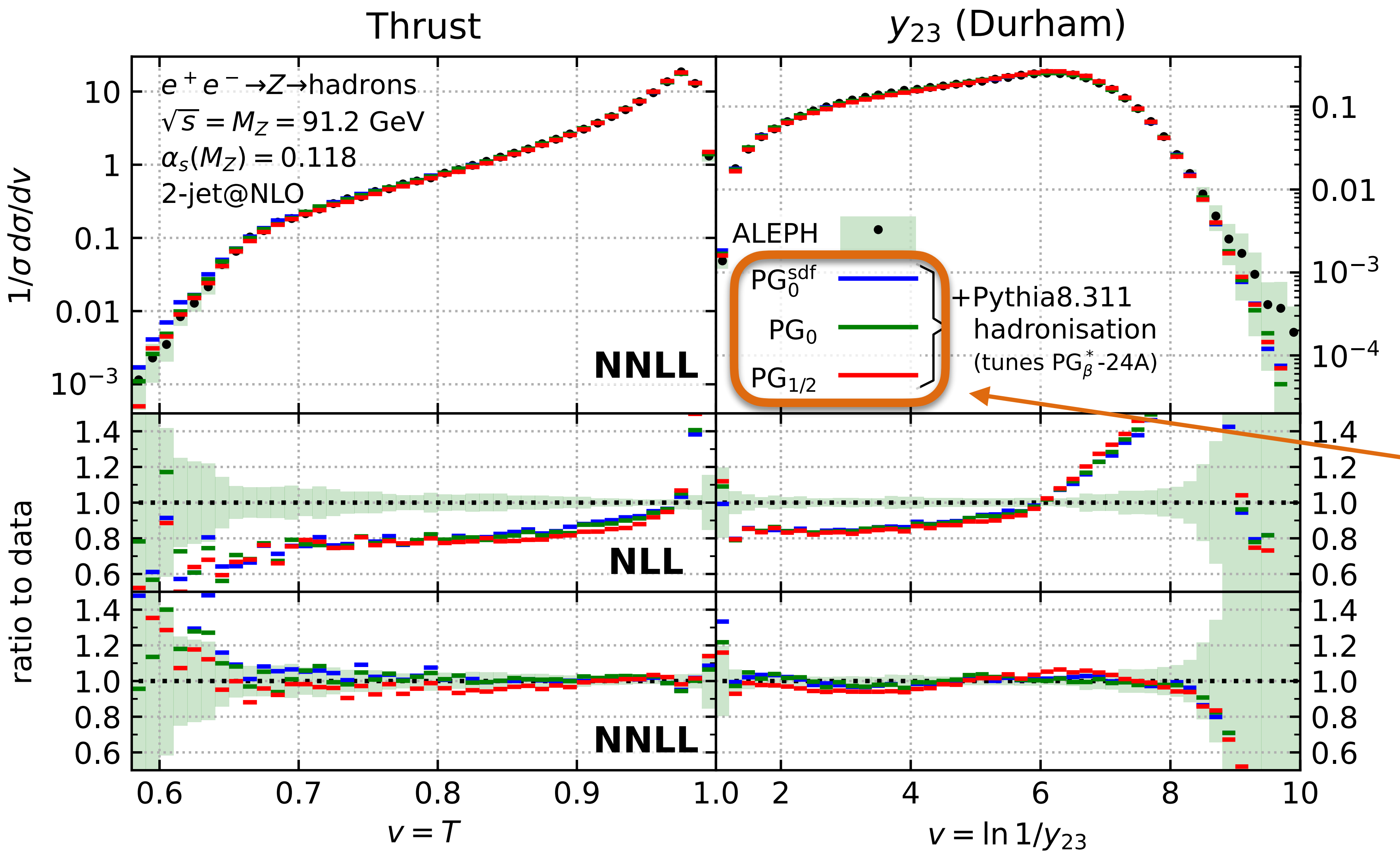
A new generation of parton showers is currently being developed which correctly resum NLL logarithms

- ALARIC, Deductor, PanScales, Herwig7, ...

The PanScales collaboration has even presented results for some observables at NNLL accuracy.

Exciting and important new development!

NNLL results for e^+e^- collisions

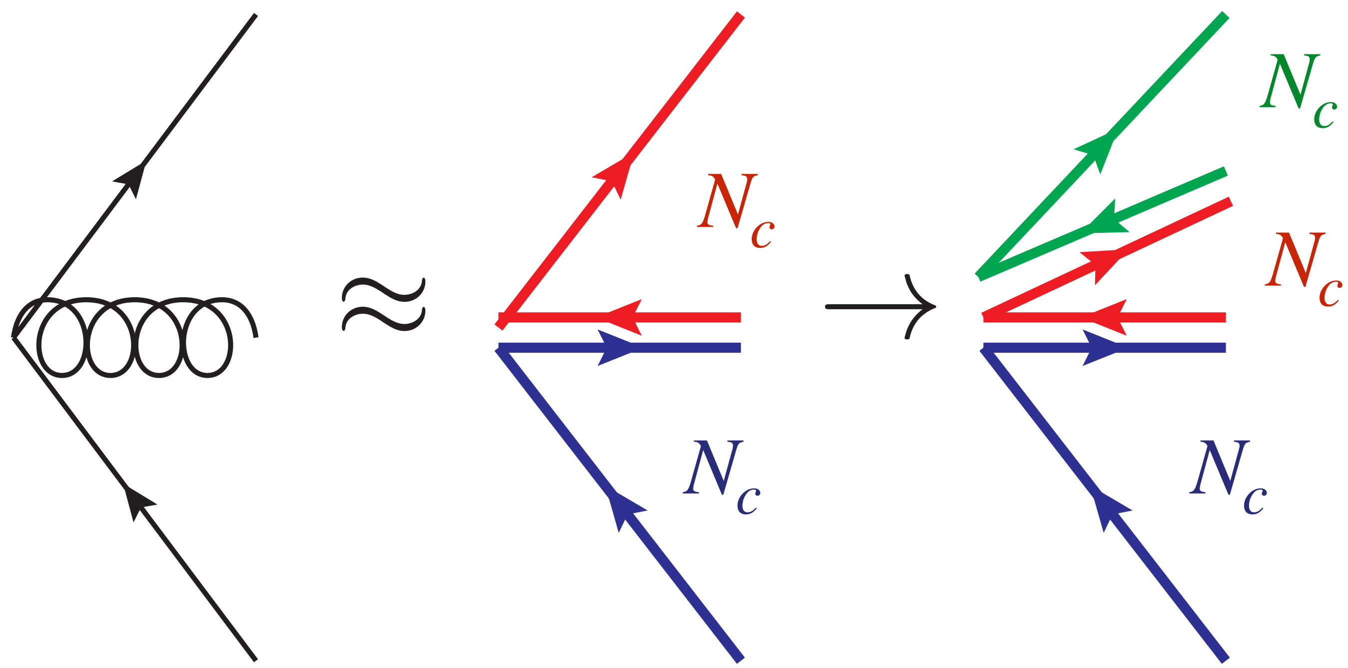


PanScales
2406.02661

different recoil schemes

- Detailed numerical checks against "analytical" resummations to verify NNLL accuracy.
- NNLL achieves marked improvement over NLL!

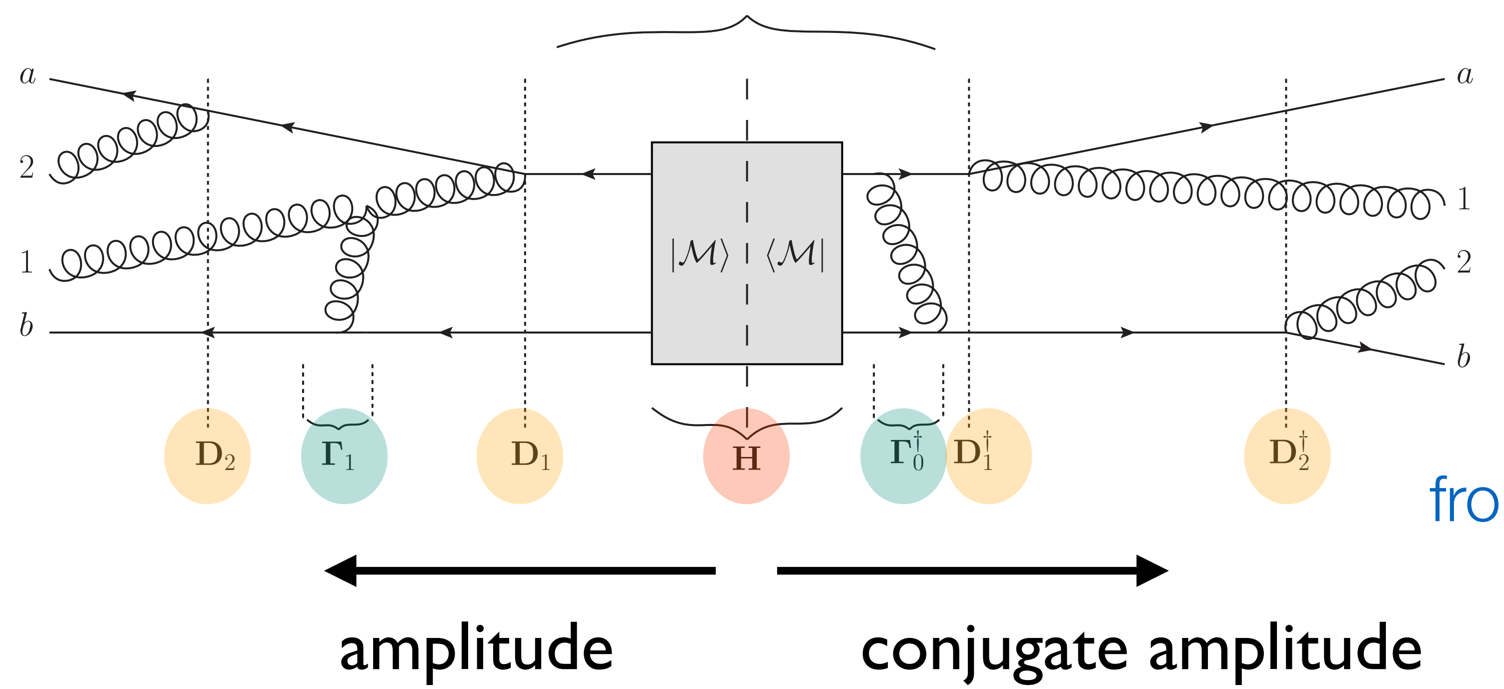
Large-N_c limit



Traditionally, parton showers work in the large-N_c limit

- huge simplification of color structure, everything is described in terms of color dipoles
- no interference, shower can be formulated on the level level of cross section

Amplitude-level evolution



from Simon Plätzer

Ongoing work on full color parton shower

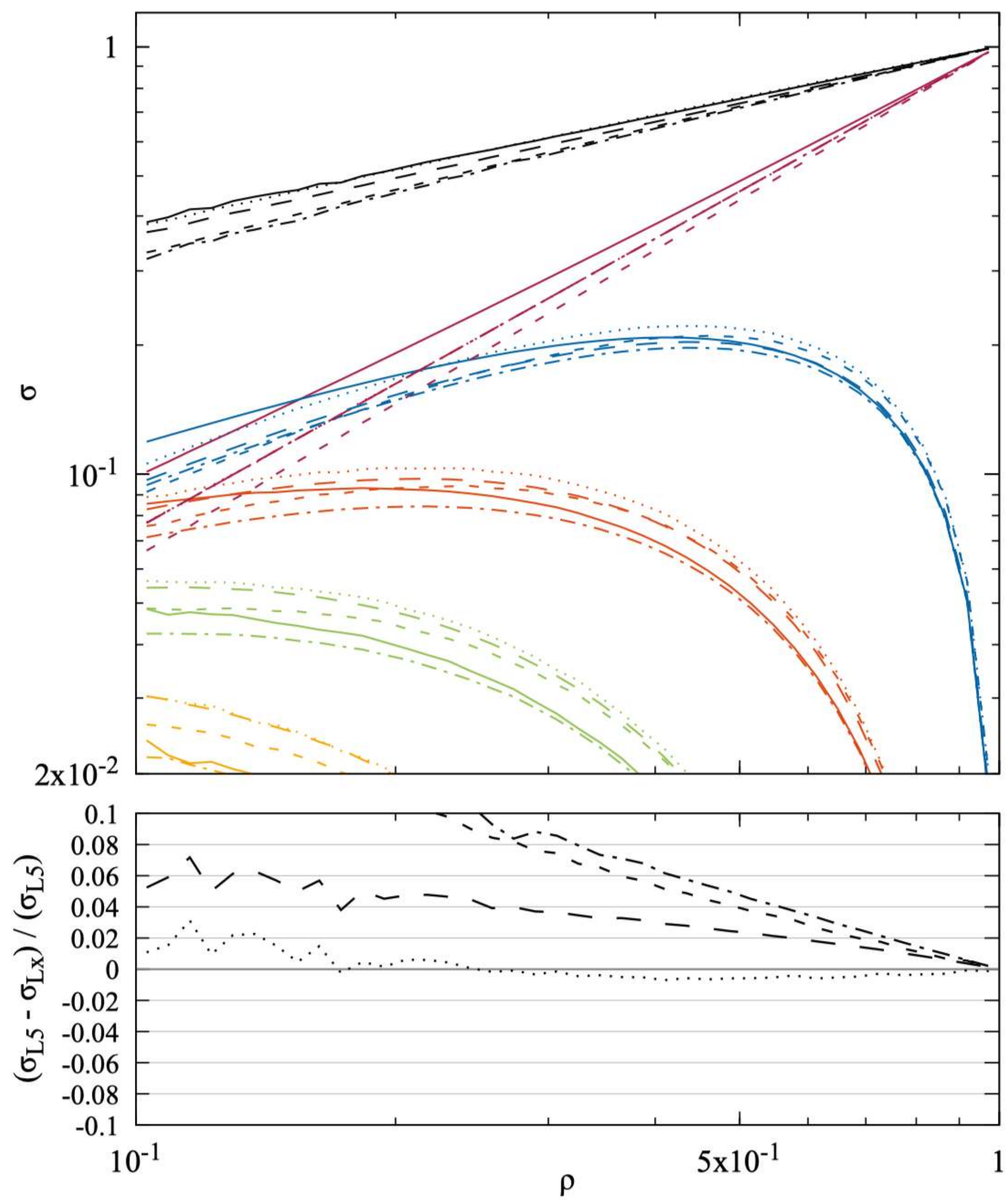
- Needs separate evolution for amplitude and its conjugate!
- Must efficiently sample the huge color space of the partons!
- Interference: no probabilistic event interpretation

Approximate treatment in `Deductor`, full color sampling in `CVolver`

CVolver results for $q\bar{q} \to q\bar{q}$

2505.13183

- Cross section with central jet veto $\rho = E_{\text{veto}}/Q$
- Partonic result only, fixed kinematics
- Plot compares full color (solid line) to strict large-N_c (short dashed) and various other approximations
- Black is full result, colors individual emissions (1 to 5)
- Agreement with results of Hatta and Ueda using Langevin method

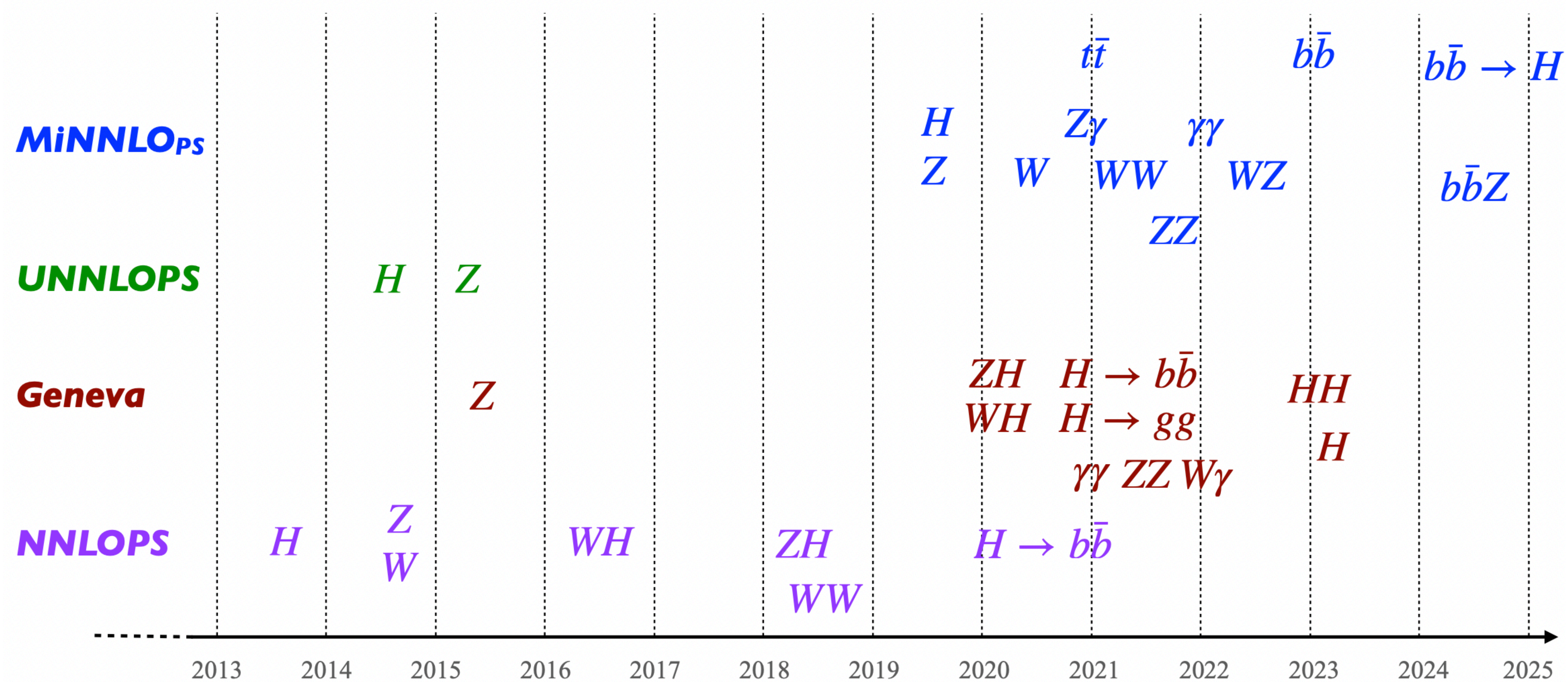


Matching to fixed order

- Shower generates emissions using approximate amplitudes (soft and collinear limits)
- Important to combine shower and fixed-order computations, so that at least the first emissions are exact
- Important to avoid double counting emissions!
- Different schemes available
  - LO (+merging): CKKW, MLM, …
  - NLO: MC@NLO, POWHEG, …
  - NNLO: MiNNLO$_{PS}$, UNNLOPS, Geneva, NNLOPS,

Parton shower matching

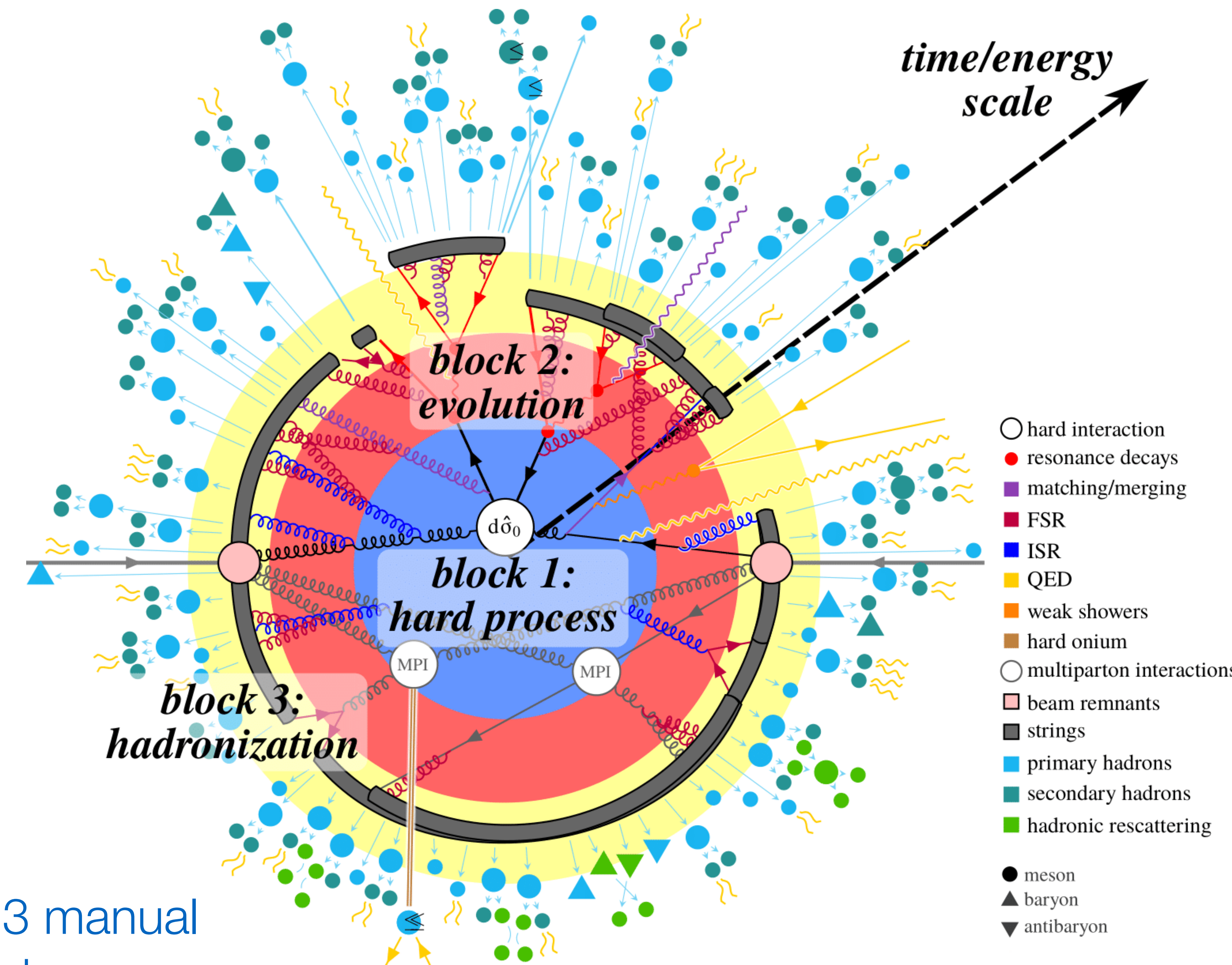
Different methods developed. NNLOPS with leading logarithmic accuracy in the shower well understood



Not yet clear how to preserve accuracy of more accurate showers in the matching

from Giulia Zanderighi's talk at Planck2025

The final frontier: hadronisation

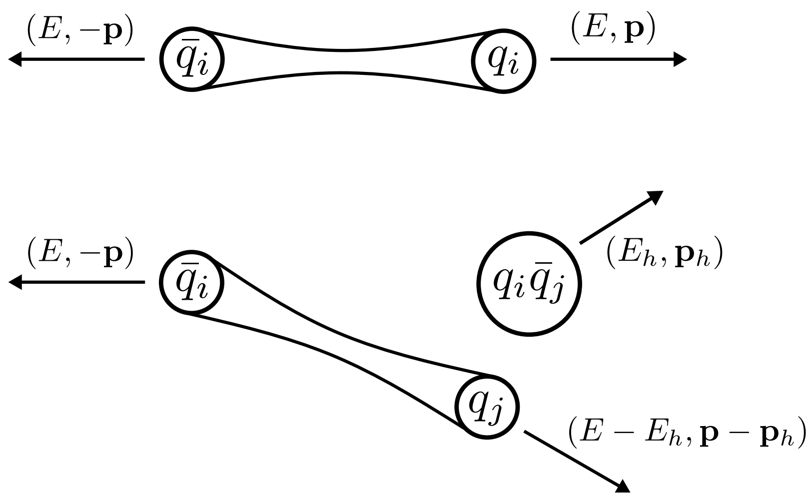


from Pythia 8.3 manual
MLHad

Hadronization models

A. Lund string model (Pythia)

Each dipole has a connecting string, hadrons through string breaking.



O(20) model parameters

B. Cluster fragmentation (Herwig)

Idea: fragmentation involves partons which are nearby in phase space.

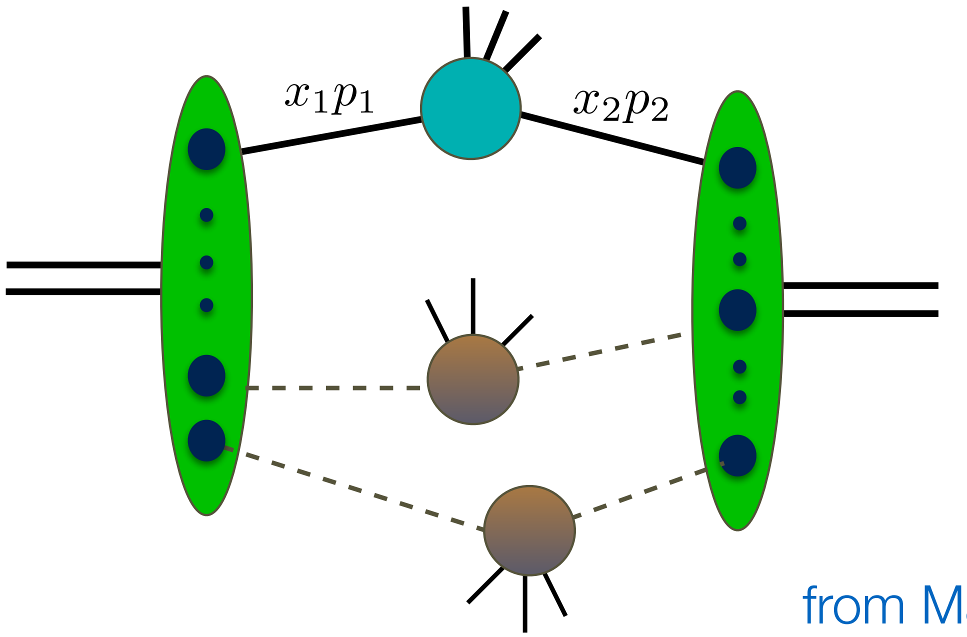
- After shower stops, form color singlet clusters of particles ("pre-confinement"). [Gluons are split into quarks.]
- Decay clusters into hadrons according to certain weights.

O(10) model parameters

Hadronisation: new developments

- HadML and MLHad: machine learning techniques to parameterize and learn hadronisation from data, estimate hadronization uncertainties
- Consistency studies with NLL showers, varying shower cutoff scale; effect on top mass? Hoang, Jin, Plätzer, Samitz
- New studies within Dokshitzer, Webber '95 model of hadronisation. Dasgupta, Hounat '24; Bafi and Farren-Colloty, Helliwell, Patel, Salam within PanScales
- Effects of color on hadronisation, within the context of amplitude showers? Plätzer, Forshaw, …
- Quantum information and hadronisation von Kuk, Lee, Michel, Sun '25

Multi-parton interactions (MPI)



from Massimiliano Grazzini

Showering and hadronizing the hard partons does not give a satisfactory description of hadron collider data.

- Shower MCs model additional collisions induced from proton remnants: MPI. ``Underlying event”
- Also include ``color reconnections” with partons from hard shower.

Conclusion

QCD is the essence of hadron collider physics!

- Understanding of QCD effects is essential for LHC precision physics program…
- … but also fascinating QFT!

QCD recently celebrated its 50th anniversary

- Mature and well developed…
- … but also many new ideas, breakthroughs (and open problems)!